Optimal trade execution and absence of price manipulations in limit order book models

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Abstract

We analyze the existence of price manipulation and optimal trade execution strategies in a model for an electronic limit order book with nonlinear price impact and exponential resilience. Our main results show that, under general conditions on the shape function of the limit order book, placing deterministic trade sizes at trading dates that are homogeneously spaced is optimal within a large class of adaptive strategies with arbitrary trading dates. This extends results from our earlier work with A. Fruth. Perhaps even more importantly, our analysis yields as a corollary that our model does not admit price manipulation strategies. This latter result contrasts the recent findings of Gatheral [12], where, in a related but different model, exponential resilience was found to give rise to price manipulation strategies when price impact is nonlinear.

1 Introduction.

The problem of optimal trade execution is concerned with the optimal acquisition or liquidation of large asset positions. In doing so, it is usually beneficial to split up the large order into a sequence of partial orders, which are then spread over a certain time horizon, so as to reduce the overall price impact and the trade execution costs. The optimization problem at hand is thus to find a trading strategy that minimizes a cost criterion under the constraint of overall order trade execution within a given time frame. There are several reasons why studying this problem is interesting.

First, liquidity risk is one of the least understood sources of financial risk, and one of its various aspects is the risk resulting from price impact created by trading large positions. Due to the nonlinear feedback effects on dynamic trading strategies, market impact risk is probably also among the most fascinating aspects of liquidity risk for mathematicians. The optimal trade execution problem allows studying market impact risk in its purest form. Moreover, the results obtained for this problem can serve as building blocks in a realistic analysis of more complex problems such as the hedging of derivatives in illiquid markets.

Second, the mathematical analysis of optimal trade execution strategies can help in the ongoing search for viable market impact models. As argued by Huberman and Stanzl [14] and Gatheral [12], any reasonable market impact model should not admit *price manipulation strategies* in the sense that there are no 'round trips' (i.e., trading strategies with zero balance in shares), whose expected trading costs are negative. Since every round trip can be regarded as the execution of a zero-size order, a solution of the optimal trade execution problem also includes an analysis of price manipulation strategies in the model (at least as limiting case when the order size tends to zero).

In recent years, the problem of optimal trade execution was considered for various market impact models and cost functions by authors such as Bertsimas and Lo [9], Almgren and Chriss [5, 6], Almgren [4], Obizhaeva and Wang [16], Almgren and Lorenz [7], Schied and Schöneborn [19], Schied, Schöneborn, and Tehranchi [18], and our joint papers with A. Fruth [1, 2], to mention only a few.

Here, we continue our analysis from [1, 2]. Instead of focussing on the two-sided limit order book model in [2], our emphasis is now on a zero-spread market impact model that is obtained from the one in [2] by collapsing the bid-ask spread. There are several advantages from introducing this model. First, it is easier to analyze than the two-sided model, while it still allows to transfer results to the two-sided framework;[1] see Section 2.6. Second, most other market impact models in the literature, such as those suggested by Almgren and Chriss [5, 6] or Gatheral [12], do not include a bid-ask spread. Therefore, these models and their features can be compared much better to our zero-spread model than to the two-sided model. Third, it is easier to detect model irregularities, such as price manipulation strategies, in the zero-spread model. In the two-sided model such irregularities may not emerge on an explicit level.

In our model, the limit order book consists of a certain distribution of limit ask orders at prices higher than the current price, while for lower prices there is a continuous distribution of limit buy orders. We consider a large trader who is placing market orders in this order book and thereby shifts prices according to the volume of available limit orders. Since the distribution of limit orders is allowed to be non-uniform, the price impact created by a market order is typically a nonlinear function of the order size. In reaction to price shocks created by market orders there is a subsequent recovery of the price within a certain time span. That is, the price evolution will exhibit a certain resilience. Thus, the price impact of a market order will neither be completely instantaneous nor entirely permanent but will decay exponentially with a time-dependent resilience rate. As in [2], we consider the following two distinct possibilities for modeling the resilience of the limit order book after a large market order: the exponential recovery of volume impact, or the exponential recovery of price impact.

This model is quite close to descriptions of price impact on limit order books found in empirical studies such as Biais *et al.* [10], Potters and Bouchaud [17], Bouchaud *et al.* [11], and Weber and Rosenow [20]. In particular, the existence of a strong resilience effect, which stems from the placement of new limit orders close to the bid-ask spread, seems to be a well established fact, although its quantitative features seem to be the subject of an ongoing discussion.

While in [2] we considered only strategies whose trades are placed at equidistant times, we now allow the trading times to be stopping times. This problem description is more natural than prescribing *a priori* the dates at which trading may take place. It is also more realistic than the idealization of trading in continuous time. In addition, the time-inhomogeneous description allows us to account for time-varying liquidity and thus in particular for the well-known U-shape patterns in intraday market parameters; see, e.g., [15].

[1] This technique was already used in [1, 2].

Optimal trade execution in this extended framework leads to the problem of optimizing simultaneously over both trading times and sizes. This problem is more complex than the one considered in [2] and requires new arguments. Nevertheless, our main results show that the unique optimum is attained by placing the deterministic trade sizes identified in [2] at trading dates that are homogeneously spaced with respect to the average resilience rate in between trades.

As a corollary, we show that neither of the two variants of our model admits price manipulation strategies in the sense of Huberman and Stanzl [14] and Gatheral [12], provided that the shape function of the limit order book belongs to a certain class of functions (which slightly differs for each variant). This corollary is surprising in view of recent results by Gatheral [12]. There it was shown that, in a closely related but different market impact model, exponential resilience leads to the existence of price manipulation strategies as soon as price impact is nonlinear.

This paper is organized as follows. In Section 2.1 we introduce our market impact model with its two variants. The cost optimization problem is explained in Section 2.2. In Section 2.3 we state our main results for the case of a block-shaped limit order book, which corresponds to linear price impact. This special case is much simpler than the case with nonlinear price impact. We therefore give a self-contained description and proof for this case, so that the reader can gain a quick intuition on why our results are true. The proofs for the block-shaped case rely on the results from our earlier paper [1] with A. Fruth and are provided in Section 3.1. The main results for the model variant with reversion of volume impact are stated in Section 2.4. The corresponding proofs are given in Section 3.2. The results for the model variant with reversion of price impact are stated in Section 2.5, while proofs are given in Section 3.3. In Section 2.6 we explain how our results can be transferred to the case of a two-sided limit order book model.

2 Setup and main results

In this section we first introduce the two variants of our market impact model and formulate the optimization problem. We then state our results for the particularly simple case of a block-shaped limit order book. Subsequently, we formulate our theorems for each model variant individually. Finally we explain how the results for the models described in Section 2.1 can be transferred to the case of a non-vanishing bid-ask spread.

2.1 Description of the market impact models

The model variants that we consider here are time-inhomogeneous versions of the zero-spread models introduced in [2, Appendix A]. The general aim is to model the dynamics of a limit order book that is exposed to repeated market orders by a large trader, whose goal is to liquidate a portfolio of X_0 shares within a certain time period $[0,T]$. The case $X_0 > 0$ corresponds to a long position and hence to a sell program, the case $X_0 < 0$ to a buy program. Here we neglect the bid-ask spread of the limit order book, but in Section 2.6 we will explain how our results can be carried over to limit order book models with non-vanishing bid-ask spread. In these two-sided models, buy orders only impact the ask side of the limit order book and sell orders only affect the bid side. Nevertheless, we will see that the optimal strategies are the same as in the zero-spread models.

When the large trader is inactive, the dynamics of the limit order book are determined by the actions of noise traders only. We assume that the corresponding *unaffected price process* S^0 is a rightcontinuous martingale on a given filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ such that S_0^0

is $\mathbb{P}$-a.s. equal to some constant S_0. The *actual price process* S is driven by the dynamics of S^0 and by the response of the limit order book to the market orders of the large traders. The key to modeling this response is to start by describing first the *volume impact process* E. If at time t the trader places a market oder of size ξ_t, where $\xi_t > 0$ stands for a buy order and $\xi_t < 0$ for a sell order, the volume impact process jumps from E_t to

$$E_{t+} := E_t + \xi_t. \tag{1}$$

When the large trader is inactive in between market orders, E reverts back at a given rate. In our first model variant, the *Model with volume impact reversion*, we assume that the volume impact process reverts on an exponential scale with a deterministic, time-dependent rate $t \mapsto \rho_t$, called *resilience speed*. More precisely, we assume that

$$dE_t = -\rho_t E_t \, dt \tag{2}$$

while the large investor is not placing buy orders. Equations (1) and (2) determine completely the dynamics of the volume impact process E in our first model variant.

In the next step, we describe the relation between volume impact and price impact. To this end, we assume a continuous distribution of bid and ask orders away from the unaffected price S_t^0. This distribution is described by a continuous function $f : \mathbb{R} \to [0, \infty)$ that satisfies $f(x) > 0$ for a.e. x, the *shape function*. Its intuitive meaning is that the number of shares offered at price $S_t^0 + x$ is given by $f(x)\,dx$. Thus, a volume impact of E_t shares corresponds to a price impact of D_t, which is given implicitly via

$$\int_0^{D_t} f(x)\,dx = E_t.$$

By introducing the antiderivative of f,

$$F(y) := \int_0^y f(x)\,dx, \qquad y \in \mathbb{R},$$

the relation between the volume impact process E and the *price impact process* D can be expressed as follows:

$$E_t = F(D_t) \qquad \text{and} \qquad D_t = F^{-1}(E_t). \tag{3}$$

Here we have used our assumption that $f > 0$ a.e. so that F is indeed invertible. Given the price impact process D, the *actual price process* S is defined as

$$S_t = S_t^0 + D_t. \tag{4}$$

Thus, if at time t the trader places a market order of size ξ_t, then the price process jumps from S_t to

$$S_{t+} = S_t^0 + D_{t+} = S_t^0 + F^{-1}(E_t + \xi_t);$$

see Figure 1. Hence, the price impact $D_{t+} - D_t$ will be a nonlinear function of the order size ξ_t unless f is constant between D_t and D_{t+}. The choice of a shape function that is constant throughout $\mathbb{R}$ corresponds to linear market impact and to the zero-spread version of the block-shaped limit order book model of Obizhaeva and Wang [16]. This zero-spread version was introduced in [1].

Instead of an exponential resilience of the volume impact as described in (2), one can also assume an exponential reversion of the *price impact* D. This means that one has to replace (2) by

$$dD_t = -\rho_t D_t \, dt \tag{5}$$

while the large investor is not placing buy orders. The resulting model variant will be called the *Model with price impact reversion*. We now summarize the definitions of our two model variants.

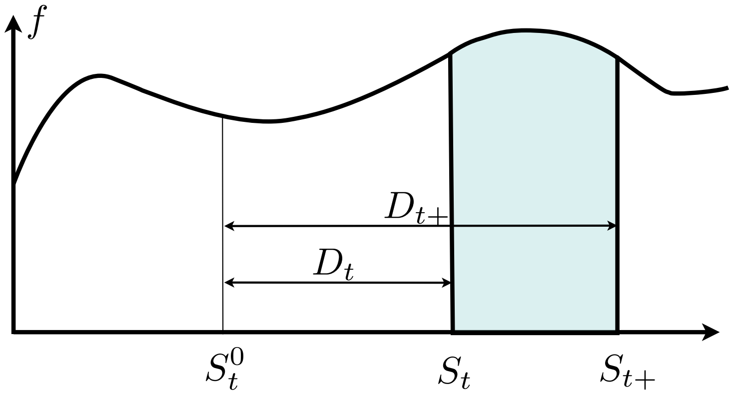

Figure 1: The price impact of a buy market order of size $\xi_t > 0$ is defined by the equation $\xi_t = \int_{D_t}^{D_{t+}} f(x)\,dx = F(D_{t+}) - F(D_t)$.

Definition 2.1. The dynamics of the *Model with volume impact reversion* are described by equations (1), (2), (3), and (4). The *Model with price impact reversion* is defined via equations (1), (5), (3), and (4).[2]

With the reversion of the processes D and E as described in Equation (2) and (5) we model the well-established empirical fact that order books exhibit a certain resilience as to the price impact of a large buy market order. That is, after the initial impact the best ask price reverts back to its previous position; cf. Biais *et al.* [10], Potters and Bouchaud [17], Bouchaud *et al.* [11], and Weber and Rosenow [20] for empirical studies.

Note that we assume that the shape function of the limit order book is neither time-dependent nor subject to additional randomness. This assumption is similar to the assumption of fixed, time-independent impact functions in the models by Bertsimas and Lo [9], Almgren [4], or Gatheral [12]. It can also be justified at least partially by the observation that the shapes of empirical limit order books for certain liquid stocks can be relatively stable over time. More importantly, it was noted in [2] that our optimal strategies are remarkably robust with respect to changes in the shape function f. One can therefore expect that a moderate randomization of f will only have relatively small effects on the optimal strategy. A corresponding analysis will be the subject of future research.

We now introduce three example classes for shape functions that satisfy the assumptions of our main results.

Example 2.2 (Block shape)**.** The simplest example corresponds to a block-shaped limit order book:

$$f(x) \equiv q \qquad \text{for some constant } q > 0. \tag{6}$$

It corresponds to the zero-spread version of the block-shaped limit order book model of Obizhaeva and Wang [16], which was introduced in [1]. In this case, the price impact function is linear: $F^{-1}(x) = x/q$. It follows that the processes D and E are related via $E_t = qD_t$, and so the model variants with volume and price impact reversion coincide. Results and proofs are particularly easy in this case. We therefore discuss it separately in Section 2.3. ◊

[2]These models are respectively referred as Model 1 and 2 in [1, 2].

Example 2.3 (Positive power-law shape)**.** Consider the class of shape functions

$$f(x) = \lambda|x|^\alpha, \qquad \text{where } \lambda,\ \alpha > 0. \tag{7}$$

In this case, $F(x) = \frac{\lambda}{1+\alpha}|x|^{1+\alpha}\operatorname{sign} x$, and so $F^{-1}(x) = (\frac{1+\alpha}{\lambda}|x|)^{1/(1+\alpha)}\operatorname{sign} x$. Hence price impact follows a power law. The choice $\alpha = 1$ corresponds to square-root impact, which is a particularly popular choice and admits certain justifications; see [12]. See also [8] for empirical results on power-law impact. We will see later that the shape functions from the class (7) satisfy the assumptions of our main results. Moreover, as was kindly pointed out to us by Jim Gatheral, our Model with volume impact reversion is equivalent to the Model with price impact reversion when we replace ρ_t by $\widetilde{\rho}_t := \frac{\rho_t}{1+\alpha}$. Indeed, when the large trader is not active during the interval $[t, t+s)$ volume impact reversion implies that $E_{t+s} = e^{-\int_t^{t+s}\rho_u\,du}E_t$. Hence,

$$D_{t+s} = F^{-1}(E_{t+s}) = e^{-\int_t^{t+s}\widetilde{\rho}_u\,du}F^{-1}(E_t) = e^{-\int_t^{t+s}\widetilde{\rho}_u\,du}D_t,$$

and so D satisfies $dD_r = -\widetilde{\rho}_r D_r\,dr$ in $[t, t+s)$. ◇

Example 2.4 (Negative power-law shape)**.** Consider the shape functions of the form

$$f(x) = \begin{cases} \dfrac{q}{(1+\lambda_+ x)^{\alpha_+}} & \text{for } x > 0, \\[2ex] \dfrac{q}{(1-\lambda_- x)^{\alpha_-}} & \text{for } x < 0, \end{cases}$$

where q and $\lambda_\pm$ are positive constants and $\alpha_\pm \in (0, 1]$. We will see later that these shape functions satisfy the assumptions of our main results. ◇

2.2 The cost optimization problem

We assume that the large trader needs to liquidate a portfolio of X_0 shares until time T and that trading can occur at $N+1$ trades within the time interval $[0, T]$. An *admissible sequence of trading times* will be a sequence $\mathcal{T} = (\tau_0, \ldots, \tau_N)$ of stopping times such that $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_N = T$. For such an admissible sequence of trading times, $\mathcal{T}$, we define a $\mathcal{T}$*-admissible trading strategy* as a sequence $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_N)$ of random variables such that

- $X_0 + \sum_{n=0}^N \xi_n = 0$ (i.e., the strategy liquidates the given portfolio X_0),
- each ξ_n is measurable with respect to $\mathcal{F}_{\tau_n}$,
- each ξ_n is bounded from below.

The quantity ξ_n corresponds to the size of the market order placed at time τ_n. Note that we do not *a priori* require that all ξ_n have the same sign, i.e., we also allow for an alternation of buy and sell orders. But we assume that there is some bound on the size of sell orders. Finally, an *admissible strategy* is a pair $(\mathcal{T}, \boldsymbol{\xi})$ consisting of an admissible sequence of trading times $\mathcal{T}$ and a $\mathcal{T}$-admissible trading strategy $\boldsymbol{\xi}$.

Let us now define the costs incurred by an admissible strategy $(\mathcal{T}, \boldsymbol{\xi})$. When at time τ_n a buy market order of size $\xi_n > 0$ is placed, the trader will purchase $f(x)\,dx$ shares at price $S^0_{\tau_n} + x$, with x ranging from D_{τ_n} to D_{τ_n+}. Hence, the total cost of the buy market order amounts to

$$\pi_{\tau_n}(\boldsymbol{\xi}) := \int_{D_{\tau_n}}^{D_{\tau_n+}} (S^0_{\tau_n} + x)f(x)\,dx = S^0_{\tau_n}\xi_n + \int_{D_{\tau_n}}^{D_{\tau_n+}} xf(x)\,dx. \tag{8}$$

Similarly, for a sell market order $\xi_n < 0$, the trader will sell $f(x)\,dx$ shares at price $S^0_{\tau_n} + x$, with x ranging from D_{τ_n+} to D_{τ_n}. Since the costs of sales should be negative, formula (8) is also valid in the case of a sell order.

The *average cost* $\mathcal{C}(\boldsymbol{\xi}, \mathcal{T})$ of an admissible strategy $(\boldsymbol{\xi}, \mathcal{T})$ is defined as the expected value of the total costs incurred by the consecutive market orders:

$$\mathcal{C}(\boldsymbol{\xi}, \mathcal{T}) = \mathbb{E}\Big[\sum_{n=0}^{N} \pi_{\tau_n}(\boldsymbol{\xi})\Big]. \tag{9}$$

The problem at hand is thus to *minimize the average cost* $\mathcal{C}(\boldsymbol{\xi}, \mathcal{T})$ *over all admissible strategies* $(\boldsymbol{\xi}, \mathcal{T})$. In doing this, we will assume for simplicity throughout this paper that the function F is unbounded in the sense that

$$\lim_{x\uparrow\infty} F(x) = \infty \quad \text{and} \quad \lim_{x\downarrow-\infty} F(x) = -\infty. \tag{10}$$

That is, we assume that the limit order book has infinite depth. Relaxing this assumption is possible but would require additional constraints on the order sizes in admissible strategies and thus complicate the problem description.

In our earlier paper with A. Fruth, [2], we considered the case of a constant resilience ρ and a fixed, equidistant time spacing $\mathcal{T}_{eq} = \{iT/N \mid i = 0, \dots, N\}$. In this setting, we determined trading strategies that minimize the cost $\mathcal{C}(\boldsymbol{\xi}, \mathcal{T}_{eq})$ among all $\mathcal{T}_{eq}$-admissible trading strategies $\boldsymbol{\xi}$. Our goal in this paper consists in *simultaneously* minimizing over trade times and sizes. Also, in our present setting of an inhomogeneous resilience function ρ_t it is natural to replace the equidistant time spacing by the *homogeneous time spacing*

$$\mathcal{T}^* = (t^*_0, \dots, t^*_N)$$

defined via

$$\int_{t^*_{i-1}}^{t^*_i} \rho_s\,ds = \frac{1}{N}\int_0^T \rho_s\,ds, \qquad i = 1, \dots, N.$$

We also define

$$a^* := e^{-\frac{1}{N}\int_0^T \rho_u du}. \tag{11}$$

Our main result states that, under certain technical assumptions, $\mathcal{T}^*$ is in fact the *unique optimal* time grid for portfolio liquidation with $N + 1$ trades in $[0, T]$. In addition, the unique optimal $\mathcal{T}^*$-admissible strategies for both model variants, i.e., for the reversion of price or volume impact, are given by the corresponding trading strategies in [2].

As a corollary to our main results, we are able to show that our models do not admit *price manipulation strategies* in the following sense, introduced by Huberman and Stanzl [14] (see also Gatheral [12]).

Definition 2.5. A *round trip* is an admissible strategy $(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}})$ for $X_0 = 0$. A *price manipulation strategy* is a round trip $(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}})$ such that $\mathcal{C}(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}}) < 0$.

Our main results also imply that our models do not possess the following model irregularity, which was introduced and discussed in our joint paper [3] with A. Slynko.

Definition 2.6. A model admits *transaction-triggered price manipulation* if the expected execution costs of a sell (buy) program can be decreased by intermediate buy (sell) trades. More precisely, there is transaction-triggered price manipulation if there exists $X_0 \in \mathbb{R}$ and a corresponding admissible strategy $(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}})$ such that

$$\mathcal{C}(\overline{\boldsymbol{\xi}}, \overline{\mathcal{T}}) < \inf \Big\{ \mathcal{C}(\boldsymbol{\xi}, \mathcal{T}) \,\Big|\, (\boldsymbol{\xi}, \mathcal{T}) \text{ is admissible and all trades in } \boldsymbol{\xi} \text{ have the same sign} \Big\}. \tag{12}$$

By taking $X_0 = 0$ in Definition 2.6, one sees that standard price manipulation in the sense of Definition 2.5 can be regarded as a special case of transaction-triggered price manipulation. It follows that the absence of transaction-triggered price manipulation implies the absence of standard price manipulation. It is possible, however, to construct models that admit transaction-triggered price manipulation but not standard price manipulation. This happens, for instance, if we take a constant shape function $f \equiv q$ and replace exponential resilience by Gaussian decay of price impact; see [3].

2.3 Main results for the block-shaped limit order book

We first discuss our problem in the particularly easy case of a block-shaped limit order book in which $f(x) = q$. In that case, our two model variants with the respective reversion of price and volume impact coincide. It follows from the results in [1] that for every admissible sequence of trading times $\mathcal{T} = (\tau_0, \dots, \tau_N)$ there is a $\mathcal{T}$-admissible trading strategy that minimizes the cost $\mathcal{C}(\cdot, \mathcal{T})$ among all $\mathcal{T}$-admissible trading strategies. This strategy can even be computed explicitly; see [1, Theorem 3.1]. In the following theorem we consider the problem of optimizing jointly over trading times and sizes.

Theorem 2.7. *In a block-shaped limit order book, there is a $\mathbb{P}$-a.s. unique optimal strategy $(\boldsymbol{\xi}^*, \mathcal{T}^*)$ consisting of homogeneous time spacing $\mathcal{T}^*$ and the deterministic trading strategy $\boldsymbol{\xi}^*$ defined by*

$$\xi_0^* = \xi_N^* = \frac{-X_0}{2 + (N-1)(1-a^*)} \qquad \textit{and} \qquad \xi_1^* = \cdots = \xi_{N-1}^* = \xi_0^*(1-a^*), \tag{13}$$

where a^* is as in (11).

While the preceding theorem is a special case of our main results, Theorem 2.11 and Theorem 2.17, it admits a particularly easy proof based on the results in [1]. This proof is given in Section 3.1.

Corollary 2.8. *In a block-shaped limit order book, there is neither standard not transaction-triggered price manipulation.*

An obvious extension of our model is to allow the resilience rate ρ_t to be a progressively measurable stochastic process. In this case, optimal strategies will look different. However, the absence of price manipulation remains valid even for this case; see Remark 3.2 at the end of Section 3.1.

Figure 2 gives an illustration of the situation when $\rho(t) = a + b\cos(t/(2\pi))$, $0 \le t \le T$. For $a > b > 0$ the resilience is greater near the opening and the closure of the stock exchange, and T represents the trade duration in days. We plot here the relative gain, i.e., the quotient of the respective expected costs, for the optimal strategies corresponding to the optimal time grid $\mathcal{T}^*$ and the equidistant time grid $\mathcal{T}_{eq}$. More precisely, we plot the quotient of the respective cost functions defined in equation (32) below.

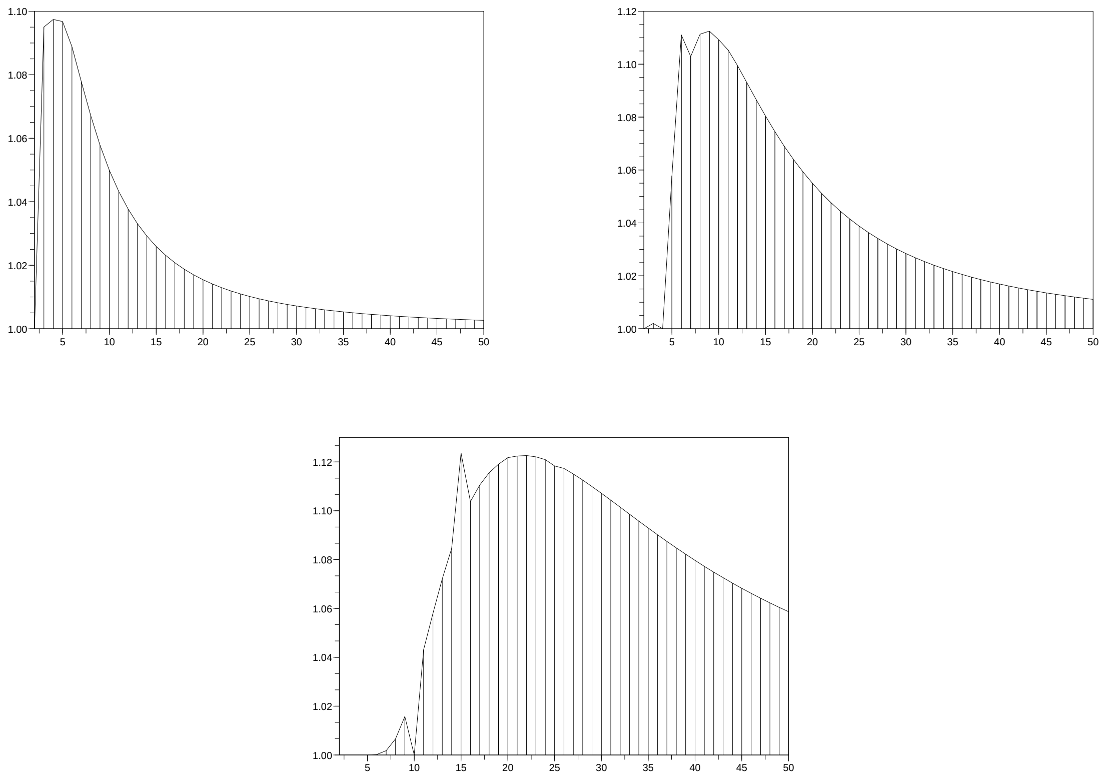

Figure 2: Relative gain between the extra liquidity cost of the optimal strategy on the optimal grid $\mathcal{T}^*$ and the optimal strategy on the equidistant grid $\mathcal{T}_{eq}$ as a function of N, when $T = 1, 2$ and 5 with the resilience function $\rho(t) = 10 + 8\cos(t/2\pi)$.

2.4 Main results for volume impact reversion

In this section we state our main results for the Model with volume impact reversion. They hold under the following assumption. Its first part covers Examples 2.2 and 2.4. Its second part covers the important case of power law price impact as introduced in Example 2.3.

Assumption 2.9. In the Model with volume impact reversion, we assume in addition to (10) that the shape function f satisfies one of the following conditions (a) and (b).

(a) f is nondecreasing on $\mathbb{R}_-$ and nonincreasing on $\mathbb{R}_+$.

(b) $f(x) = \lambda|x|^\alpha$, for constants λ, $\alpha > 0$.

We start by looking at optimal trading strategies when an admissible sequence of trading times $\mathcal{T} = (\tau_0, \ldots, \tau_N)$ is fixed. If $\boldsymbol{\xi}$ is a $\mathcal{T}$-admissible trading strategy and it happens that $\tau_i = \tau_{i+1}$, then the corresponding trades, ξ_i and ξ_{i+1}, are executed simultaneously. We therefore say that two $\mathcal{T}$-admissible trading strategies $\boldsymbol{\xi}$ and $\overline{\boldsymbol{\xi}}$ are *equivalent* if $\xi_i + \xi_{i+1} = \overline{\xi}_i + \overline{\xi}_{i+1}$ $\mathbb{P}$-a.s. on $\{\tau_i = \tau_{i+1}\}$.

Proposition 2.10. *Suppose that an admissible sequence of trading times $\mathcal{T}$ is given and that Assumption 2.9 holds. Then there exists a $\mathcal{T}$-admissible trading strategy $\boldsymbol{\xi}^{V,\mathcal{T}}$, $\mathbb{P}$-a.s. unique up to equivalence, that minimizes the cost $\mathcal{C}(\cdot, \mathcal{T})$ among all $\mathcal{T}$-admissible trading strategies. Moreover, $\boldsymbol{\xi}^{V,\mathcal{T}} \neq 0$ for $X_0 \neq 0$ up to equivalence, and all components of $\boldsymbol{\xi}^{V,\mathcal{T}}$ have the same sign. For $X_0 = 0$, all components of $\boldsymbol{\xi}^{V,\mathcal{T}}$ are zero.*

As we will see in the proof of Proposition 2.10, the optimal trading strategy $\boldsymbol{\xi}^{V,\mathcal{T}}$ can be implicitly characterized via a certain nonlinear equation. Our main result for the case of volume impact reversion states, however, that things become much easier when optimizing simultaneously over trading times and sizes:

Theorem 2.11. *Under Assumption 2.9, for every $X_0 \neq 0$ there is a $\mathbb{P}$-a.s. unique optimal strategy $(\boldsymbol{\xi}^V, \mathcal{T}^*)$ consisting of homogeneous time spacing $\mathcal{T}^*$ and the deterministic trading strategy $\boldsymbol{\xi}^V$ that is defined as follows. The initial market order ξ_0^V is the unique solution of the equation*

$$F^{-1}\big(-X_0 - N\xi_0^V\,(1-a^*)\big) = \frac{F^{-1}(\xi_0^V) - a^* F^{-1}(a^*\xi_0^V)}{1-a^*}, \tag{14}$$

the intermediate orders are given by

$$\xi_1^V = \cdots = \xi_{N-1}^V = \xi_0^V\,(1-a^*)\,, \tag{15}$$

and the final order is determined by

$$\xi_N^V = -X_0 - \xi_0^V - (N-1)\xi_0^V\,(1-a^*)\,.$$

Moreover, $\xi_0^V \neq 0$ and all components of $\boldsymbol{\xi}^V$ have the same sign. That is, $\boldsymbol{\xi}^V$ consists only of nontrivial sell orders for $X_0 > 0$ and only of nontrivial buy orders for $X_0 < 0$.

The preceding results imply the following corollary relating to Definitions 2.5 and 2.6.

Corollary 2.12. *Under Assumption 2.9, the Model with volume impact reversion admits neither standard nor transaction-triggered price manipulation.*

The absence of price manipulation stated in the preceding corollary remains valid even for the case in which resilience ρ_t is stochastic; see Remark 3.2 at the end of Section 3.1.

Corollary 2.12 shows that, in our Model with volume reversion, exponential resilience of price impact is well compatible with nonlinear impact governed by a shape function that satisfies Assumption 2.9. We thus deduce that, at least from a theoretical perspective, exponential resilience of a limit order book is a viable possibility for describing the decay of market impact. This contrasts Gatheral's [12] observation that, in a related but different continuous-time model, exponential decay of price impact gives rise to price manipulation in the sense of Definition 2.5 as soon as price impact is nonlinear. Given the strong contrasts between the results in [12] and our Corollary 2.12, it is interesting to discuss the relations between the model in [12] and our model.

Remark 2.13 (Relation to Gatheral's model)**.** In [12], a continuous-time model is introduced, which is closely related to our model. Formulating a discrete-time variant within our setting leads to the following definition for the actual price process:

$$S_t^{JG} = S_t^0 + \sum_{\tau_n < t} h(\xi_n)\psi(t-\tau_n). \tag{16}$$

Here, $h : \mathbb{R} \to \mathbb{R}$ is the *price impact function*, and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is the *decay kernel*. The decay kernel describes the time decay of price impact. If we take $\psi(t) = e^{-\rho t}$ for a constant $\rho > 0$, (16) takes the form

$$S_t^{JG} = S_t^0 + \sum_{\tau_n < t} h(\xi_n)e^{-\rho(t-\tau_n)}. \tag{17}$$

It was shown in Section 4 of [12] that the continuous-time version of (17) admits price manipulation in the sense of Definition 2.5 as soon as the function h is not linear. By approximating a continuous-time price manipulation strategy with discrete-time strategies, one sees that this result carries over to the discrete-time framework (17).

When taking a constant resilience speed ρ in our Model with volume impact reversion, the volume impact process is of the form $E_t = \sum_{\tau_n<t} \xi_n e^{-\rho(t-\tau_n)}$, and so our price process is given by

$$S_t = S_t^0 + F^{-1}\Big(\sum_{\tau_n<t} \xi_n e^{-\rho(t-\tau_n)}\Big). \tag{18}$$

Thus, the difference between the two models is that in (17) the nonlinear price impact function is applied to each individual trade, while in (18) the price impact is obtained as a nonlinear function of the volume impact. Therefore, both models are different, and there is no contradiction between the results in [12] and Corollary 2.12. ◇

Remark 2.14 (Continuous-time limit)**.** Let us briefly discuss the asymptotic behavior of the optimal strategy when the number N of trades tends to infinity. Since for any N we have that ξ_0^V lies strictly between 0 and $-X_0$, we can extract a subsequence that converges to some $\xi_0^{V,\infty}$. One therefore checks that the right-hand side of (14) tends to

$$h_V^\infty(\xi_0^{V,\infty}) := F^{-1}(\xi_0^{V,\infty}) + \frac{\xi_0^{V,\infty}}{f(F^{-1}(\xi_0^{V,\infty}))}.$$

Since $N(1-a^*) \to \int_0^T \rho_s\,ds$, the left-hand side of (14) converges as well, and so $\xi_0^{V,\infty}$ must be a solution y of the equation

$$F^{-1}\Big(-X_0 - y\int_0^T \rho_s\,ds\Big) = h_V^\infty(y).$$

Note that, under our assumptions, h_V^∞ is strictly increasing. Hence, the preceding equation has a unique solution, which consequently must be the limit of ξ_0^V as $N \uparrow \infty$. It follows moreover that $N\xi_1^V \to \xi_0^{V,\infty}\int_0^T \rho_s\,ds$ and that

$$\xi_N^V \longrightarrow -X_0 - \xi_0^{V,\infty} - \xi_0^V \int_0^T \rho_s\,ds =: \xi_T^{V,\infty}.$$

Thus, the optimal strategy, described in "resilience time" $r(t) := \int_0^t \rho_s\,ds$, consists of an initial block trade of size $\xi_0^{V,\infty}$, continuous buying at constant rate $\xi_0^{V,\infty}$ during $(0,T)$, and a final block trade of size $\xi_T^{V,\infty}$. Transforming back to standard time leaves the initial and final block trades unaffected, and continuous buying in $(0,T)$ now occurs at the time-dependent rate $\rho_t\xi_0^{V,\infty}$.

One can expect that this limiting strategy could be optimal in the following continuous-time variant of our model, which is similar to the setup in [13]. A strategy is a predictable processes $t \mapsto X_t$ of bounded total variation, which describes the number of shares in the portfolio of the trader at time t. Given such a strategy, the process E is defined via $E_0 = 0$ and

$$dE_t = dX_t - \rho_t E_t\,dt, \tag{19}$$

and D is given by $D_t = F(E_t)$. The optimal strategy obtained above then corresponds to

$$dX_t^* = \xi_0^{V,\infty}\delta_0(dt) + \xi_0^{V,\infty}\rho_t\,dt + \xi_T^{V,\infty}\delta_T(dt). \qquad \diamondsuit$$

2.5 Main results for reversion of price impact

In this section we state our main results for the Model with reversion of price impact. This case is analytically more complicated than the Model with volume impact, because the quantity that decays exponentially is no longer a linear function of the order size. We therefore need a stronger assumption:

Assumption 2.15. In addition to (10) we assume that the shape function f satisfies one of the following conditions (a) and (b).

(a) f is twice differentiable on $\mathbb{R}\backslash\{0\}$, nondecreasing on $\mathbb{R}_-$ and nonincreasing on $\mathbb{R}_+$, and satisfies

$$x \mapsto xf'(x)/f(x) \text{ is nondecreasing on } \mathbb{R}_-, \text{ nonincreasing on } \mathbb{R}_+, \text{ and } (-1,0]\text{-valued}, \tag{20}$$

$$1 + x\frac{f'(x)}{f(x)} + 2x^2\left(\frac{f'(x)}{f(x)}\right)^2 - x^2\frac{f''(x)}{f(x)} \geq 0 \qquad \text{for all } x \geq 0. \tag{21}$$

(b) $f(x) = \lambda|x|^\alpha$, for constants λ, $\alpha > 0$.

We will see in Example 2.19 below that Assumption 2.15 (a) is satisfied for the power law shape functions from Example 2.4. Also, we know already from Example 2.3 that under Assumption 2.15 (b) that the Model with reversion of price impact is equivalent to a model with reversion of volume impact.

We start by looking at optimal trading strategies when an admissible sequence of trading times $\mathcal{T} = (\tau_0, \dots, \tau_N)$ is fixed. As in Section 2.4, we say that two $\mathcal{T}$-admissible trading strategies $\boldsymbol{\xi}$ and $\overline{\boldsymbol{\xi}}$ are equivalent if $\xi_i + \xi_{i+1} = \overline{\xi}_i + \overline{\xi}_{i+1}$ $\mathbb{P}$-a.s. on $\{\tau_i = \tau_{i+1}\}$.

Proposition 2.16. *Suppose that an admissible sequence of trading times $\mathcal{T}$ is given and that Assumption 2.15 holds. Then there exists a $\mathcal{T}$-admissible trading strategy $\boldsymbol{\xi}^{P,\mathcal{T}}$, $\mathbb{P}$-a.s. unique up to equivalence, that minimizes the cost $\mathcal{C}(\cdot, \mathcal{T})$. Moreover, $\boldsymbol{\xi}^{P,\mathcal{T}} \neq 0$ for $X_0 \neq 0$ up to equivalence, and all components of $\boldsymbol{\xi}^{P,\mathcal{T}}$ have the same sign. For $X_0 = 0$, all components of $\boldsymbol{\xi}^{P,\mathcal{T}}$ are zero.*

As in Proposition 2.10, computing the optimal trading strategy $\boldsymbol{\xi}^{P,\mathcal{T}}$ for an arbitrary sequence $\mathcal{T}$ can be quite complicated. But again the structure becomes much easier when optimizing also over the sequence of trading times $\mathcal{T}$. To state the corresponding result, let us recall from (11) the definition of a^* and let us introduce the function

$$h_{P,a^*}(x) := x \frac{f(x/a^*)/a^* - a^* f(x)}{f(x/a^*) - a^* f(x)}.$$

We will see in Lemma 3.8 (a) below that $h_{P,a^*}(x)$ is indeed well-defined for all $x \in \mathbb{R}$ as soon as Assumption 2.15 is satisfied.

Theorem 2.17. *Suppose that that the shape function f satisfies Assumption 2.15. Then for $X_0 \neq 0$ there is a $\mathbb{P}$-a.s. unique optimal strategy $(\boldsymbol{\xi}^P, \mathcal{T}^*)$, consisting of homogeneous time spacing $\mathcal{T}^*$ and the deterministic trading strategy $\boldsymbol{\xi}^P$ that is defined as follows. The initial market order ξ_0^P is the unique solution of the equation*

$$F^{-1}\left(-X_0 - N\left[\xi_0^P - F\left(a^* F^{-1}(\xi_0^P)\right)\right]\right) = h_{P,a^*}\left(F^{-1}(\xi_0^P)\right), \tag{22}$$

the intermediate orders are given by

$$\xi_1^P = \dots = \xi_{N-1}^P = \xi_0^P - F\left(a^* F^{-1}(\xi_0^P)\right), \tag{23}$$

and the final order is determined by

$$\xi_N^P = -X_0 - N\xi_0^P + (N-1)F\left(a^* F^{-1}(\xi_0^P)\right).$$

Moreover, $\xi_0^P \neq 0$ and all components of $\boldsymbol{\xi}^P$ have the same sign. That is, $\boldsymbol{\xi}^P$ consists only of nontrivial sell orders for $X_0 > 0$ and only of nontrivial buy orders for $X_0 < 0$.

Again, the preceding results lead to the following corollary. Its conclusion on the absence of price manipulation remains valid even for the case in which resilience ρ_t is stochastic; see Remark 3.2 at the end of Section 3.1.

Corollary 2.18. *Under Assumption 2.15, the Model with price impact reversion admits neither standard nor transaction-triggered price manipulation.*

We continue this section by showing that the power law shape functions from Example 2.4 satisfy Assumption 2.15 (a).

Example 2.19 (Negative power-law shape)**.** Let us show that the power-law shape functions from Example 2.4 satisfy our Assumption 2.15 (a). For checking (20) and (21) we concentrate on the branch of f on the positive part of the real line. So let us suppose that

$$f(x) = \frac{q}{(1+\lambda x)^\alpha}, \qquad \text{for } x > 0,$$

with $\alpha \in [0,1]$, $q, \lambda > 0$. We have $xf'(x)/f(x) = -\frac{\alpha\lambda x}{1+\lambda x} \in (-1, 0]$ which is nonincreasing on $\mathbb{R}_+$. Moreover, for $x \geq 0$ we have

$$1 + x\frac{f'(x)}{f(x)} + 2x^2\left(\frac{f'(x)}{f(x)}\right)^2 - x^2\frac{f''(x)}{f(x)} = \frac{1 + (2-\alpha)\lambda x + (1 - 2\alpha + \alpha^2)(\lambda x)^2}{(1+\lambda x)^2} \geq 0. \qquad \diamondsuit$$

Remark 2.20. With a positive power-law shape, we know from Example 2.3 that the model with volume impact reversion ρ_t is equivalent to the model with price impact reversion $\tilde{\rho}_t = \rho_t/(1+\alpha)$. Thus, the strategy defined via (14) and (15) with a^* is the same as the one defined by (22) and (23) with $\tilde{a}^* = (a^*)^{\frac{1}{1+\alpha}}$. This can be checked by a straightforward calculation. $\diamondsuit$

Remark 2.21. As in Remark 2.14, we can study the asymptotic behavior of the optimal strategy as the number N of trades tends to infinity. First, one checks that h_{P,a^*} converges to

$$h_P^\infty(x) := x\Big(1 + \frac{f(x)}{f(x) + xf'(x)}\Big),$$

and that $N(y - F(a^* F^{-1}(y)))$ tends to $F^{-1}(y) f(F^{-1}(y)) \int_0^T \rho_s \, ds$. Now, suppose that the equation

$$F^{-1}\Big(-X_0 - F^{-1}(y) f(F^{-1}(y)) \int_0^T \rho_s \, ds\Big) = h_P^\infty(F^{-1}(y))$$

has a unique solution that lies strictly between 0 and $-X_0$, and which we will call $\xi_0^{P,\infty}$. We then see as in Remark 2.14 that $\xi_0^{P,\infty}$ is the limit of ξ_0^P when $N \uparrow \infty$. Next, $N\xi_1^P$ converges to $F^{-1}(\xi_0^{P,\infty}) f(F^{-1}(\xi_0^{P,\infty})) \int_0^T \rho_s \, ds$ and ξ_N^P to

$$\xi_T^{P,\infty} := -X_0 - \xi_0^{P,\infty} - F^{-1}(\xi_0^{P,\infty}) f(F^{-1}(\xi_0^{P,\infty})) \int_0^T \rho_s \, ds.$$

This yields a description of the continuous-time limit in "resilience time" $r(t) := \int_0^t \rho_s \, ds$. Using a time change as in Remark 2.14, we obtain that the optimal strategy consists of an initial block order of $\xi_0^{P,\infty}$ shares at time 0, continuous buying at rate $\rho_t F^{-1}(\xi_0^{P,\infty}) f(F^{-1}(\xi_0^{P,\infty}))$ during $(0, T)$, and a final block order of $\xi_T^{P,\infty}$ shares at time T. One might guess that this strategy should be optimal in the continuous-time model in which strategies are predictable processes $t \mapsto X_t$ of total bounded variation and the volume impact process satisfies

$$dE_t = dX_t - \rho_t g(E_t) \, dt$$

for $g(x) = f(F^{-1}(x)) F^{-1}(x)$; see also Remark 2.14. $\diamondsuit$

2.6 Two-sided limit order book models

We now explain how our results can be used to analyze models for an electronic limit order book with a nonvanishing and dynamic bid-ask spread. To this end, we focus on a buy program with $X_0 < 0$ (the case of a sell program is analogous). Therefore emphasis is on buy orders, and we concentrate first on the upper part of the limit order book, which consists of shares offered at various ask prices. The lowest ask price at which shares are offered is called the *best ask price*. When the large trader is inactive, the dynamics of the limit order book are determined by the actions of noise traders only. We assume that the corresponding *unaffected best ask price* A^0 is a rightcontinuous martingale on a given filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ and satisfies $A^0_0 = A_0$ $\mathbb{P}$-a.s. for some constant A_0. Above the unaffected best ask price A^0_t, we assume a continuous distribution for ask limit orders: the number of shares offered at price $A^0_t + x$ with $x \geq 0$ is given by $f(x)\,dx$, where f is a given shape function.

The *actual best ask price* at time t, i.e., the best ask price after taking the price impact of previous buy orders of the large trader into account, is denoted by A_t. It lies above the unaffected best ask price, and the *price impact* on ask prices caused by the actions of the large trader is denoted by

$$D^A_t := A_t - A^0_t.$$

A buy market order of $\xi_t > 0$ shares placed by the large trader at time t will consume all the ask limit orders offered at prices between A_t and $A_{t+} := A^0_t + D^A_{t+}$, where D^A_{t+} is determined by the condition $\int_{D^A_t}^{D^A_{t+}} f(x)dx = \xi_t$. Thus, the process D^A captures only the impact of market buy orders on the current best ask price. We also define the *volume impact* on ask orders by $E^A_t := F(D^A_t)$, where again $F(x) = \int_0^x f(y)\,dy$.

On the bid side of the limit order book, we have an *unaffected best bid process*, B^0_t. All we assume on its dynamics is $B^0_t \leq A^0_t$ at all times t. The distribution of bids below B^0_t is modeled by the restriction of the shape function f to the domain $(-\infty, 0)$. In analogy to the ask part, we introduce the the *price impact* on bid prices by $D^B_t := B_t - B^0_t$. The process D^B will be nonpositive. A sell market order of $\xi_t < 0$ shares placed at time t will consume all the shares offered at prices between B_t and $B_{t+} := B^0_t + D^B_{t+}$, where D^B_{t+} is determined by the condition

$$\xi_t = F(D^B_{t+}) - F(D^B_t) =: E^B_{t+} - E^B_t,$$

for $E^B_s := F(D^B_s)$. As before, there are now two distinct variants for modeling the reversion of the volume and price impact processes while the trader is not active. More precisely, we assume that

$$\begin{array}{lclcl} dE^A_t = -\rho_t E^A_t\,dt & \text{and} & dE^B_t = -\rho_t E^B_t\,dt & & \text{for reversion of volume impact} \\ dD^A_t = -\rho_t D^A_t\,dt & \text{and} & dD^B_t = -\rho_t D^B_t\,dt & & \text{for reversion of price impact.} \end{array} \tag{24}$$

The respective model variants will be called the *two-sided limit order book models with reversion of volume* or *price impact*.

Finally, we define the costs of an admissible strategy $(\mathcal{T}, \boldsymbol{\xi})$. We can argue as in Section 2.2 that the costs incurred at time τ_n should be defined as

$$\overline{\pi}_{\tau_n}(\boldsymbol{\xi}) = \begin{cases} \displaystyle\int_{D^A_{\tau_n}}^{D^A_{\tau_n+}} f(x)\,dx & \text{for } \xi_n > 0, \\ 0 & \text{for } \xi_n = 0, \\ \displaystyle\int_{D^B_{\tau_n}}^{D^B_{\tau_n+}} f(x)\,dx & \text{for } \xi_n < 0. \end{cases} \tag{25}$$

The following result compares the costs in the two-sided model to the costs $\pi_{\tau_n}(\boldsymbol{\xi})$ defined in (8).

Proposition 2.22. *Suppose that $A^0 = S^0$. Then, for any strategy $\boldsymbol{\xi}$, we have $\overline{\pi}_{\tau_n}(\boldsymbol{\xi}) \geq \pi_{\tau_n}(\boldsymbol{\xi})$ for all n, with equality if all trades in $\boldsymbol{\xi}$ are nonnegative.*

The preceding result can be proved by arguments given in [2, Section A]. Together with the observation that there is no transaction-triggered price manipulation in the models introduced in Section 2.1, it provides the key to transferring results on the basic model to the order book model. We thus have the following corollary.

Corollary 2.23. *Suppose that $A^0 = S^0$.*

(a) *Under Assumption 2.9, the strategy $(\boldsymbol{\xi}^V, \mathcal{T}^*)$ defined in Theorem 2.11 is the unique optimal strategy in the two-sided limit order book model with volume impact reversion.*

(b) *Under Assumption 2.15, the strategy $(\boldsymbol{\xi}^P, \mathcal{T}^*)$ defined in Theorem 2.17 is the unique optimal strategy in the two-sided limit order book model with price impact reversion.*

3 Proofs

In a first step, note that the average costs introduced in (9) are of the form

$$\mathcal{C}(\boldsymbol{\xi}, \mathcal{T}) = \mathbb{E}\Big[\sum_{n=0}^{N} \pi_{\tau_n}(\boldsymbol{\xi})\Big] = \mathbb{E}\Big[\sum_{n=0}^{N} \xi_n S^0_{\tau_n}\Big] + \mathbb{E}\Big[\sum_{n=0}^{N} \int_{D_{\tau_n}}^{D_{\tau_n+}} x f(x)\, dx\Big].$$

Due to the martingale property of S^0, optional stopping, and the fact that $\sum_{n=0}^N \xi_n = -X_0$, the first expectation on the right is equal to $-X_0 S_0$. Next, note that the process D evolves deterministically once the values of $\tau_0(\omega), \dots, \tau_N(\omega)$ and $\xi_0(\omega), \dots, \xi_N(\omega)$ are given. Thus, when the functional $\sum_{n=0}^N \int_{D_{\tau_n}}^{D_{\tau_n+}} x f(x)\, dx$ admits a unique deterministic minimizer, this minimizer must be equal to the unique optimal strategy.

To formulate the resulting deterministic optimization problem, it will be convenient to work with the quantities

$$\alpha_k := \int_{\tau_{k-1}}^{\tau_k} \rho_s ds, \qquad k = 1, \dots, N, \tag{26}$$

instead of the τ_k themselves. The condition $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_N = T$ is clearly equivalent to $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_N)$ belonging to

$$\mathcal{A} := \Big\{\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N_+ \,\Big|\, \sum_{k=1}^N \alpha_k = \int_0^T \rho_s\, ds\Big\}.$$

By abuse of notation, we will write

$$E_n \text{ and } D_n \text{ instead of } E_{\tau_n} \text{ and } D_{\tau_n} \tag{27}$$

as long as there is no possible confusion. We will also write

$$E_{n+} = E_n + \xi_n \text{ and } D_{n+} = D_n + \xi_n \text{ instead of } E_{\tau_n+} \text{ and } D_{\tau_n+}. \tag{28}$$

Then

$$\begin{aligned} E_{k+1} &= e^{-\alpha_{k+1}} E_{k+} = e^{-\alpha_{k+1}} (E_k + \xi_k) && \text{for volume impact reversion,} \\ D_{k+1} &= e^{-\alpha_{k+1}} D_{k+} = e^{-\alpha_{k+1}} F^{-1} \left(\xi_k + F\left(D_k\right)\right) && \text{for price impact reversion.} \end{aligned} \tag{29}$$

With this notation, it follows that there exist two deterministic functions C^V, $C^P : \mathbb{R}^{N+1} \times \mathcal{A} \to \mathbb{R}$ such that

$$\sum_{n=0}^{N} \int_{D_n}^{D_{n+}} x f(x)\, dx = \begin{cases} C^V(\boldsymbol{\xi}, \boldsymbol{\alpha}) & \text{in the Model with volume impact reversion,} \\ C^P(\boldsymbol{\xi}, \boldsymbol{\alpha}) & \text{in the Model with price impact reversion.} \end{cases} \tag{30}$$

We will show in the respective Sections 3.2 and 3.3 that, under our assumptions, the functions C^V and C^P have unique minima within the set $\Xi \times \mathcal{A}$, where

$$\Xi := \Big\{ \boldsymbol{x} = (x_0, \dots, x_N) \in \mathbb{R}^{N+1} \,\Big|\, X_0 + \sum_{n=0}^{N} x_n = 0 \Big\}.$$

When working with deterministic trading strategies in Ξ rather than with random variables, we will mainly use Roman letters like $\boldsymbol{x}$ instead of Greek letters such as $\boldsymbol{\xi}$. Last, we introduce the functions

$$\tilde{F}(x) := \int_0^x z f(z)\, dz \qquad \text{and} \qquad G = \tilde{F} \circ F^{-1}. \tag{31}$$

We conclude this section with the following easy lemma.

Lemma 3.1. *For $X_0 < 0$, there is no $\boldsymbol{x} \in \Xi$ such that $E_{n+} = E_n + x_n \leq 0$ (or, equivalently, $D_{n+} \leq 0$) for all $n = 0, \dots, N$.*

Proof: Since the effect of resilience is to drive the extra spread back to zero, we have $E_{n+} \geq x_0 + \cdots + x_n$ up to and including the first n at which $x_0 + \cdots + x_n > 0$. Since $x_0 + \cdots + x_N = -X_0 > 0$, the result follows. □

3.1 Proofs for a block-shaped limit order book

In this section, we give quick and direct proofs for our results in case of a block-shaped limit order book with $f(x) = q$. In this setting, our two model variants coincide; see Example 2.2. As explained in [1], the cost function in (30) is an increasing affine function of

$$C(\boldsymbol{x}, \boldsymbol{\alpha}) = \frac{1}{2} \langle \boldsymbol{x}, M(\boldsymbol{\alpha}) \boldsymbol{x} \rangle, \qquad \boldsymbol{x} \in \Xi,\ \alpha \in \mathcal{A}, \tag{32}$$

where $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product and $M(\boldsymbol{\alpha})$ is the positive definite symmetric matrix with entries

$$M(\boldsymbol{\alpha})_{n,m} = \exp\Big(- \int_{\tau_{n \wedge m}}^{\tau_{n \vee m}} \rho_u du \Big) = \exp\Big(- \Big| \sum_{i=1}^{n} \alpha_i - \sum_{j=1}^{m} \alpha_j \Big| \Big), \qquad 0 \leq n, m \leq N.$$

Proof of Theorem 2.7. For $\boldsymbol{\alpha}$ belonging to

$$\mathcal{A}^* := \big\{ \boldsymbol{\alpha} \in \mathcal{A} \,|\, \alpha_i > 0,\ i = 1, \dots, N \big\},$$

the inverse $M(\boldsymbol{\alpha})^{-1}$ of the matrix $M(\boldsymbol{\alpha})$ can be computed explicitly, and the unique optimal trading strategy for fixed $\boldsymbol{\alpha}$ is

$$\boldsymbol{x}^*(\boldsymbol{\alpha}) = \frac{-X_0}{\langle \mathbf{1}, M(\boldsymbol{\alpha})^{-1}\mathbf{1}\rangle} M(\boldsymbol{\alpha})^{-1}\mathbf{1}.$$

By [1, Theorem 3.1] the vector $M(\boldsymbol{\alpha})^{-1}\mathbf{1}$ has only strictly positive components for $\boldsymbol{\alpha} \in \mathcal{A}^*$. It follows that

$$\begin{aligned}
\min_{\boldsymbol{x}\in\Xi} C(\boldsymbol{x}, \boldsymbol{\alpha}) &= C(\boldsymbol{x}^*(\boldsymbol{\alpha}), \boldsymbol{\alpha}) = \frac{X_0^2}{2\langle \mathbf{1}, M(\boldsymbol{\alpha})^{-1}\mathbf{1}\rangle} \\
&= \frac{X_0^2}{2}\left(\frac{2}{1+e^{-\alpha_1}} + \sum_{n=2}^{N} \frac{1-e^{-\alpha_n}}{1+e^{-\alpha_n}}\right)^{-1} \\
&= \frac{X_0^2}{2}\left(\sum_{n=1}^{N} \frac{2}{1+e^{-\alpha_n}} - (N-1)\right)^{-1}.
\end{aligned} \tag{33}$$

Minimizing $\min_{\boldsymbol{x}\in\Xi} C(\boldsymbol{x}, \boldsymbol{\alpha})$ over $\boldsymbol{\alpha} \in \mathcal{A}^*$ is thus equivalent to maximizing $\sum_{n=1}^{N} \frac{2}{1+e^{-\alpha_n}}$. The function $a \mapsto \frac{2}{1+e^{-a}}$ is strictly concave in $a > 0$. Hence,

$$\sum_{n=1}^{N} \frac{2}{1+e^{-\alpha_n}} \leq \frac{2N}{1+e^{-\frac{1}{N}\sum_{n=1}^{N}\alpha_n}} = \frac{2N}{1+e^{-\frac{1}{N}\int_0^T \rho_u du}},$$

with equality if and only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}^*$, where $\boldsymbol{\alpha}^*$ corresponds to to homogeneous time spacing $\mathcal{T}^*$, i.e.,

$$\alpha_i^* = \frac{1}{N}\int_0^T \rho_s \, ds, \qquad i = 1, \dots, N. \tag{34}$$

Next, $C(\boldsymbol{x}, \boldsymbol{\alpha})$ is clearly jointly continuous in $\boldsymbol{x} \in \Xi$ and $\boldsymbol{\alpha} \in \mathcal{A}$, so $\inf_{\boldsymbol{x}\in\Xi} C(\boldsymbol{x}, \boldsymbol{\alpha})$ is upper semicontinuous in $\boldsymbol{\alpha}$. One thus sees that the minimum cannot be attained at the boundary of $\mathcal{A}$. Finally, the formula (13) for the optimal trading strategy with homogeneous time spacing can be found in [1, Remark 3.2] or in [2, Corollary 6.1]. □

Proof of Corollary 2.8. The result follows immediately from Theorem 2.7. □

Remark 3.2 (Stochastic resilience)**.** Suppose that the resilience rate ρ_t is not necessarily deterministic but can also be progressively measurable stochastic process. We assume moreover that ρ_t is strictly positive and integrable. Then the expected costs of any admissible strategy $(\mathcal{T}, \boldsymbol{\xi})$ will still be of the form

$$\mathcal{C}(\boldsymbol{\xi}, \mathcal{T}) = \mathbb{E}[\, C(\boldsymbol{\xi}, \boldsymbol{\alpha})\,].$$

Since $C(\boldsymbol{\xi}, \boldsymbol{\alpha}) \geq 0$ for every round trip by Corollary 2.8, the same is true for $\mathcal{C}(\boldsymbol{\xi}, \mathcal{T})$, and hence there cannot be price manipulation for any round trip, even under stochastic resilience. The same argument also applies in the contexts of Sections 2.4 and 2.5. ◇

3.2 Proofs for reversion of volume impact

We need a few lemmas before we can prove our main results for reversion of volume impact. For simplicity, we will first prove our results for $X_0 < 0$. The case for $X_0 > 0$ will then follow by the symmetry of the problem formulation. The case of round trips with $X_0 = 0$ will be analyzed by a limiting procedure in the proof of Proposition 2.10.

The subsequent Lemma 3.3 formulates analytic properties implied by our Assumption 2.9. In fact, only these properties will be needed in the remainder of the proof, and so our results remain valid for all shape functions f that satisfy the *conclusions* of Lemma 3.3. The next step in our proof is to prove existence and uniqueness of optimal strategies. This is done in Lemma 3.6 by exploiting the fact that, in the Model with volume impact reversion, the cost functional is coercive and strictly convex. Note, however, that convexity will be lost in the Model with price impact reversion. The final and most delicate step of the proof is to show that the first-order condition for optimality yields a nonlinear equation that uniquely determines the optimal strategy. The recursive formula of Lemma 3.4 is a preliminary step into that direction.

Lemma 3.3. *Under Assumption 2.9, the following conclusions hold.*

(a) *For each $a \in (0,1)$, the function $h_{V,a}(y) = F^{-1}(y) - aF^{-1}(ay)$ is strictly increasing on $\mathbb{R}$.*

(b) *For all $a, b \in (0,1)$ and $\nu > 0$, we have the inequalities:*

$$h_{V,a}^{-1}\big(\nu(1-a)\big) > b \cdot h_{V,b}^{-1}\big(\nu(1-b)\big), \tag{35}$$

$$b \cdot h_{V,b}^{-1}\big(\nu(1-b)\big) < F(\nu). \tag{36}$$

(c) *The function $H_V : (y,a) \in (0,\infty) \times (0,1) \mapsto \Big(\frac{F^{-1}(y)-aF^{-1}(ay)}{1-a}, ay\frac{F^{-1}(y)-F^{-1}(ay)}{1-a}\Big) \in \mathbb{R}^2$ is one-to-one.*

Proof of Lemma 3.3 under Assumption 2.9 (a): Assumption 2.9 (a) states that f is increasing on $\mathbb{R}_-$ and decreasing on $\mathbb{R}_+$. Part (a) of the assertion thus follows from [2, Remark 2].

For the proof of part (b), let $y := h_{V,a}^{-1}\big(\nu(1-a)\big)$. Then $y > 0$ since $h_{V,a}(0) = 0$. Note also that F^{-1} is convex on $\mathbb{R}_+$. Let $\widehat{f}$ be its derivative. Then,

$$\begin{aligned}
\nu &= \frac{F^{-1}(y) - aF^{-1}(ay)}{1-a} = F^{-1}(ay) + \frac{F^{-1}(y) - F^{-1}(ay)}{1-a} \\
&= F^{-1}(ay) + \frac{1}{1-a}\int_{ay}^{y} \widehat{f}(x)\,dx < F^{-1}(y) + y\widehat{f}(y) =: g(y).
\end{aligned}$$

Clearly, g is a strictly increasing function on $\mathbb{R}_+$, and so we have $y > g^{-1}(\nu)$.

Next, let $z := b \cdot h_{V,b}^{-1}\big(\nu(1-b)\big)$. Then,

$$\begin{aligned}
\nu &= \frac{F^{-1}(z/b) - bF^{-1}(z)}{1-b} = F^{-1}(z) + \frac{F^{-1}(z/b) - F^{-1}(z)}{1-b} \\
&= F^{-1}(z) + \frac{1}{1-b}\int_{z}^{z/b} \widehat{f}(x)\,dx \geq F^{-1}(z) + z\widehat{f}(z) = g(z),
\end{aligned}$$

since $\widehat{f}(z) = 1/f(F^{-1}(z))$ is nondecreasing for $z > 0$. Thus, $z \leq g^{-1}(\nu) < h_{V,a}^{-1}\big(\nu(1-a)\big)$, and (35) follows. For (36) it now suffices to note that $g(z) > F^{-1}(z)$.

To prove part (c), let $a_1, a_2 \in (0,1)$ and $y_1, y_2 > 0$ and assume that $H_V(a_1, y_1) = H_V(a_2, y_2)$. Since

$$\frac{F^{-1}(y) - aF^{-1}(ay)}{1-a} = F^{-1}(y) + a\frac{F^{-1}(y) - F^{-1}(ay)}{1-a},$$

we get

$$\begin{aligned} F^{-1}(y_1) + a_1 \frac{F^{-1}(y_1) - F^{-1}(a_1 y_1)}{1-a_1} &= F^{-1}(y_2) + a_2 \frac{F^{-1}(y_2) - F^{-1}(a_2 y_2)}{1-a_2}, \\ a_1 y_1 \frac{F^{-1}(y_1) - F^{-1}(a_1 y_1)}{1-a_1} &= a_2 y_2 \frac{F^{-1}(y_2) - F^{-1}(a_2 y_2)}{1-a_2}. \end{aligned} \tag{37}$$

Assume that $y_1 \neq y_2$, say, $y_1 > y_2 > 0$. Multiplying the first identity by y_1 and subtracting the second identity yields

$$y_1 \frac{F^{-1}(y_1) - F^{-1}(y_2)}{y_1 - y_2} = \frac{a_2}{1-a_2}\big[F^{-1}(y_2) - F^{-1}(a_2 y_2)\big]. \tag{38}$$

Since $(F^{-1})'(y) = \widehat{f}(y)$ is nondecreasing for $y > 0$, we obtain that

$$y_1 \frac{F^{-1}(y_1) - F^{-1}(y_2)}{y_1 - y_2} \geq y_1 \widehat{f}(y_2) \quad \text{and} \quad \frac{a_2}{1-a_2}\big[F^{-1}(y_2) - F^{-1}(a_2 y_2)\big] \leq a_2 y_2 \widehat{f}(y_2),$$

which contradicts the previous equation since $y_1 > y_2 \geq a_2 y_2$. Therefore we must have $y_1 = y_2$.

It is therefore sufficient to show that

$$\widetilde{h}(a) := \frac{a}{1-a}\big[F^{-1}(y) - F^{-1}(ay)\big], \qquad a \in]0,1[,$$

is one-to-one for any $y > 0$. Its derivative is equal to

$$\widetilde{h}'(a) = \frac{1}{(1-a)^2}\big[F^{-1}(y) - F^{-1}(ay)\big] - \frac{ay}{1-a}\widehat{f}(ay). \tag{39}$$

Using again that $\widehat{f}(y)$ is nondecreasing for $y > 0$, we get

$$F^{-1}(y) - F^{-1}(ay) > (1-a) y \widehat{f}(ay)$$

and in turn $\widetilde{h}'(a) > 0$. $\square$

Proof of Lemma 3.3 under Assumption 2.9 (b): Assumption 2.9 (b) states that $f(x) = \lambda |x|^\alpha$ for constants $\lambda, \alpha > 0$. Thus,

$$h_{V,a}(y) = \Big(\frac{1+\alpha}{\lambda}\Big)^{\frac{1}{1+\alpha}} \big(1 - a^{\frac{2+\alpha}{1+\alpha}}\big) |y|^{\frac{1}{1+\alpha}} \operatorname{sign} y, \tag{40}$$

and so part (a) of the assertion follows.

As for part (b) of the assertion, note that

$$h_{V,a}^{-1}\big(\nu(1-a)\big) = \frac{\lambda}{1+\alpha} \nu^{1+\alpha} \left(\frac{1-a}{1-a^{\frac{2+\alpha}{1+\alpha}}}\right)^{1+\alpha}$$

Hence, when a goes from 0 to 1, the value of $h_{V,a}^{-1}(\nu(1-a))$ decreases from $\frac{\lambda}{1+\alpha}\nu^{1+\alpha}$ to $\frac{\lambda}{1+\alpha}\nu^{1+\alpha}\big(\frac{1+\alpha}{2+\alpha}\big)^{1+\alpha}$. Using the shorthand notation $\gamma := 1 + \alpha$, the inequality (35) will thus follow if we can show that

$$b^{\frac{1}{\gamma}} \frac{1-b}{1-b^{\frac{1+\gamma}{\gamma}}} < \frac{\gamma}{1+\gamma} \tag{41}$$

for $0 < b < 1$ and $\gamma > 1$. The preceding inequality is equivalent to

$$1 - b < \frac{\gamma}{1+\gamma}(b^{-1/\gamma} - b).$$

But the function on the right is a strictly convex decreasing function of b, whose derivative at $b = 1$ is -1. This proves the asserted inequalities and in turn (35). Inequality (36) is obvious, given our formulas for $h_{V,b}^{-1}\big(\nu(1-b)\big)$ and F.

To prove (c), we use the same argument as under Assumption 2.9 (a). First, let us observe that $\Psi : x \in \mathbb{R}_+ \mapsto x(x^{1/(1+\alpha)} - 1)/(x-1)$ is increasing, which can be easily checked by derivating. If $y_1 \neq y_2$, say, $z = y_1/y_2 > 1$, we get $\Psi(z) = \Psi(a_2)$ from (38) which is not possible since $z > 1 > a_2$. Thus $y_1 = y_2$, and we get from (37) that $\Psi(a_1) = \Psi(a_2)$, which gives $a_1 = a_2$. □

Let us turn to the calculation of the cost derivatives. With (31), the cost function (30) in the Model with volume impact reversion can be represented as

$$C^V(\boldsymbol{x}, \boldsymbol{\alpha}) = \sum_{n=0}^{N} \Big[G(E_n + x_n) - G(E_n)\Big], \qquad \boldsymbol{x} \in \Xi,\ \boldsymbol{\alpha} \in \mathcal{A}, \tag{42}$$

where

$$E_0 = 0 \qquad \text{and} \qquad E_n = \sum_{i=0}^{n-1} x_i e^{-\sum_{k=i+1}^{n} \alpha_k}, \qquad 1 \le n \le N.$$

Lemma 3.4. *For $i = 0, \dots, N-1$, we have the following recursive formula,*

$$\frac{\partial C^V}{\partial x_i} = F^{-1}(E_i + x_i) - e^{-\alpha_{i+1}} F^{-1}(E_{i+1}) + e^{-\alpha_{i+1}} \frac{\partial C^V}{\partial x_{i+1}}. \tag{43}$$

Moreover, for $i = 1, \dots, N$,

$$\frac{\partial C^V}{\partial \alpha_i} = E_i \sum_{n=i}^{N} \big[F^{-1}(E_n + x_n) - F^{-1}(E_n)\big] e^{-\sum_{k=i+1}^{n} \alpha_k}. \tag{44}$$

Proof: To prove (43), we first need to calculate $\partial E_n / \partial x_i$. We obtain:

$$\frac{\partial E_n}{\partial x_i} = 0 \qquad \text{if } i \ge n, \text{ and} \qquad \frac{\partial E_n}{\partial x_i} = e^{-\sum_{k=i+1}^{n} \alpha_k} \qquad \text{if } i < n.$$

Using the fact that $G' = F^{-1}$, we therefore get

$$\begin{aligned}
\frac{\partial C^V}{\partial x_i} &= F^{-1}(E_i + x_i) + \sum_{n=i+1}^{N} \big[F^{-1}(E_n + x_n) - F^{-1}(E_n)\big] e^{-\sum_{k=i+1}^{n} \alpha_k} \\
&= F^{-1}(E_i + x_i) - e^{-\alpha_{i+1}} F^{-1}(E_{i+1}) \\
&\quad + e^{-\alpha_{i+1}} \left(F^{-1}(E_{i+1} + x_{i+1}) + \sum_{n=i+2}^{N} \big[F^{-1}(E_n + x_n) - F^{-1}(E_n)\big] e^{-\sum_{k=i+2}^{n} \alpha_k} \right),
\end{aligned}$$

which yields (43).

For the proof of (44), we have first to compute $\partial E_n/\partial \alpha_i$. We obtain:

$$\frac{\partial E_n}{\partial \alpha_i} = 0 \qquad \text{if } i > n, \text{ and} \qquad \frac{\partial E_n}{\partial \alpha_i} = -\sum_{m=0}^{i-1} x_m e^{-\sum_{k=m+1}^{n} \alpha_k} \qquad \text{for } i \leq n.$$

From here, we get

$$\begin{aligned} \frac{\partial C^V}{\partial \alpha_i} &= -\sum_{n=i}^{N} \big[F^{-1}(E_n + x_n) - F^{-1}(E_n)\big] \sum_{m=0}^{i-1} x_m e^{-\sum_{k=m+1}^{n} \alpha_k} \\ &= E_i \sum_{n=i}^{N} \big[F^{-1}(E_n + x_n) - F^{-1}(E_n)\big] e^{-\sum_{k=i+1}^{n} \alpha_k}, \end{aligned}$$

which is (44). □

Remark 3.5. A consequence of this lemma is that homogeneous time spacing $\boldsymbol{\alpha}^*$ and the optimal strategy $\boldsymbol{\xi}^V$ given in [2] yield a critical point for the minimization in $(\boldsymbol{x}, \boldsymbol{\alpha})$. Indeed, we have then $E_i = a^* \xi_0^V$ for any i, and therefore $\frac{\partial C^V}{\partial \alpha_i}$ does not depend on i. ♢

Lemma 3.6. *For each $\boldsymbol{\alpha} \in \mathcal{A}$ the function $C^V(\cdot, \boldsymbol{\alpha})$ has a minimizer $\boldsymbol{x}^*(\boldsymbol{\alpha}) \in \Xi$, which is unique up to equivalence.*

Proof: First note that we may assume without loss of generality that $\boldsymbol{\alpha} \in \mathcal{A}^* = \{\boldsymbol{\alpha} \in \mathcal{A} \,|\, \alpha_i > 0,\ i = 1, \ldots, N\}$. Indeed, if $\alpha_i = 0$ we can merge the trades x_{i-1} and x_i into a single one and reduce N to $N-1$.

We next extend the arguments in [2, Lemma B.1] to prove the existence of a unique minimizer of $C^V(\cdot, \boldsymbol{\alpha})$ in Ξ.

Using the convention $\sum_{k=n+1}^{n} \alpha_k := 0$, we obtain by rearranging the sum in (42) that

$$\begin{aligned} C^V(\boldsymbol{x}, \boldsymbol{\alpha}) &= G\Big(\sum_{i=0}^{N} x_i e^{-\sum_{k=i+1}^{N} \alpha_k}\Big) - G(0) \\ &\quad + \sum_{n=0}^{N-1} \Big[G\Big(\sum_{i=0}^{n} x_i e^{-\sum_{k=i+1}^{n} \alpha_k}\Big) - G\Big(e^{-\alpha_{n+1}} \sum_{i=0}^{n} x_i e^{-\sum_{k=i+1}^{n} \alpha_k}\Big)\Big]. \end{aligned}$$

Let us define the linear map $T : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ via

$$(T\boldsymbol{x})_n = \sum_{i=0}^{n} x_i e^{-\sum_{k=i+1}^{n} \alpha_k}, \qquad n = 0, \ldots, N.$$

We can thus write

$$C^V(\boldsymbol{x}, \boldsymbol{\alpha}) = G\big((T\boldsymbol{x})_N\big) - G(0) + \sum_{n=0}^{N-1} \Big[G\big((T\boldsymbol{x})_n\big) - G\big(e^{-\alpha_{n+1}} (T\boldsymbol{x})_n\big)\Big]. \tag{45}$$

Note first that G is strictly convex since $G' = F^{-1}$ is strictly increasing. Second, for $a \in (0,1)$, the function $x \to G(x) - G(ax)$ is also strictly convex, because its derivative is equal to the strictly increasing function $h_{V,a}$ in Lemma 3.3. And third, T is one-to-one. Hence, $C^V(\cdot, \boldsymbol{\alpha})$ is strictly convex in is first argument, and there can be at most one minimizer.

To show the existence of a minimizer, note that $G' = F^{-1}$ is increasing with $F^{-1}(0) = 0$, and hence $G(y) - G(ay) \geq (1-a)|y| \cdot |F^{-1}(ay)|$. Therefore, (45) yields

$$\begin{aligned} C^V(\boldsymbol{x}, \boldsymbol{\alpha}) \;\; &\geq \;\; G\big((T\boldsymbol{x})_N\big) - G(0) \\ &\qquad + \sum_{n=0}^{N-1} (1 - e^{-\alpha_{n+1}}) \cdot \big|F^{-1}\big(e^{-\alpha_{n+1}}(T\boldsymbol{x})_n\big)\big| \cdot \Big|(T\boldsymbol{x})_n\Big|. \end{aligned}$$

Hence,

$$C^V(\boldsymbol{x}, \boldsymbol{\alpha}) \geq \Lambda\big(|T\boldsymbol{x}|_\infty\big) - G(0),$$

where $|\cdot|_\infty$ is the ℓ^∞-norm on $\mathbb{R}^{N+1}$ and Λ is the function

$$\Lambda(y) := G(y) \wedge G(-y) \wedge \min_{n=0,\ldots,N-1} \Big\{ |y| \cdot (1 - a_{n+1}) \Big(\big|F^{-1}(a_{n+1} \cdot y)\big| \wedge \big|F^{-1}(-a_{n+1} \cdot y)\big| \Big) \Big\},$$

where $a_{n+1} := e^{-\alpha_{n+1}}$. Since F is unbounded, both $G(y)$ and $|F^{-1}(y)|$ tend to $+\infty$ for $|y| \to \infty$, and the fact that T is one-to-one implies that $\Lambda\big(|T\boldsymbol{x}|_\infty\big) \to +\infty$ for $|\boldsymbol{x}| \to \infty$. Note also that by assumption $\alpha_n > 0$ for each n. Hence, $C^V(\cdot, \boldsymbol{\alpha})$ must attain its minimum on Ξ. $\square$

We are now in a position to prove the main results for the Model with reversion of volume impact.

Proof of Proposition 2.10: The result for $X_0 < 0$ will follow if we can show that the minimizer in Lemma 3.6 consists only of strictly positive components. Here, we may assume without loss of generality that the admissible sequence of trading times is strictly increasing, or equivalently that $\boldsymbol{\alpha} \in \mathcal{A}^*$, for otherwise we can simply merge two trades occurring at the same time into a single trade.

If $\boldsymbol{x}$ is the minimizer of $C^V(\cdot, \boldsymbol{\alpha})$ on Ξ, then there must be a Lagrange multiplier ν such that $\boldsymbol{x}$ is a critical point of $\boldsymbol{y} \mapsto C^V(\boldsymbol{y}, \boldsymbol{\alpha}) - \nu \sum_{i=0}^N y_i$. Hence, (43) yields that

$$\nu(1 - a_{i+1}) = F^{-1}(E_i + x_i) - a_{i+1}F^{-1}(E_{i+1}) = h_{V,a_{i+1}}(E_i + x_i), \qquad i = 0, \ldots, N-1, \tag{46}$$

where $a_{i+1} = e^{-\alpha_{i+1}}$ and $h_{V,a}$ is as in Lemma 3.3. For the final trade, we have

$$\nu = F^{-1}(E_N + x_N). \tag{47}$$

Since $h_{V,a}(0) = 0 = F^{-1}(0)$ and both $h_{V,a}$ and F^{-1} are strictly increasing, we conclude that $E_0 + x_0, \ldots, E_N + x_N$ have all the same sign as ν. Thus, $\nu > 0$ by Lemma 3.1. Next, (46) implies that $E_i + x_i = h^{-1}_{V,a_{i+1}}(\nu(1 - a_{i+1}))$ and hence $E_{i+1} = a_{i+1}h^{-1}_{V,a_{i+1}}(\nu(1 - a_{i+1}))$. Using (46) once again yields

$$x_0 = h^{-1}_{V,a_1}(\nu(1 - a_1)) \quad \text{and} \quad x_i = h^{-1}_{V,a_{i+1}}(\nu(1 - a_{i+1})) - a_i h^{-1}_{V,a_i}(\nu(1 - a_i)), \qquad i = 1, \ldots, N-1.$$

The inequality (35) thus gives $x_i > 0$ for $i = 0, \ldots, N-1$. As for the final trade, (47) gives $x_N = F(\nu) - a_N h^{-1}_{V,a_N}(\nu(1 - a_N))$, which is strictly positive by (36).

Now we consider the case $X_0 = 0$. Suppose that $(\boldsymbol{x}, \boldsymbol{\alpha})$ is a round trip such that $C^V(\boldsymbol{x}, \boldsymbol{\alpha}) \leq 0$. Again we can assume w.l.o.g. that $\boldsymbol{\alpha} \in \mathcal{A}^*$. Then

$$C^V(\boldsymbol{x}, \boldsymbol{\alpha}) = \lim_{\varepsilon \downarrow 0} C^V(\boldsymbol{x} + \varepsilon \mathbf{1}, \boldsymbol{\alpha}).$$

But $C^V(\boldsymbol{x}+\varepsilon\mathbf{1}, \boldsymbol{\alpha}) > 0$ for each $\varepsilon > 0$, due to our previous results. Hence we must have $C^V(\boldsymbol{x}, \boldsymbol{\alpha}) = 0$. The strict convexity of $C^V(\cdot, \boldsymbol{\alpha})$, established in the proof of Lemma 3.6, implies that there can be at most one minimizer of $C^V(\cdot, \boldsymbol{\alpha})$ in the class of round trips. Since we clearly have $C^V(\mathbf{0}, \boldsymbol{\alpha}) = 0$, we must conclude that $\boldsymbol{x} = \mathbf{0}$. □

Proof of Theorem 2.11: We will show that for $X_0 < 0$ the admissible strategy $(\boldsymbol{\xi}^V, \boldsymbol{\alpha}^*)$, defined via (14), (15), and (34), is the unique minimizer of C^V on $\Xi \times \mathcal{A}$. The first step is to show the existence of a minimizer. To this end, note that Proposition 2.10 allows us to restrict the minimization of C^V to $\Xi_+ \times \mathcal{A}$, where $\Xi_+ = \{\boldsymbol{x} \in \Xi \,|\, x_i \geq 0,\ i = 0, \dots, N\}$. The set $\Xi_+ \times \mathcal{A}$ is in fact the product of two compact simplices, and so the continuity of C^V yields the existence of a global minimizer, which lies in $\Xi_+ \times \mathcal{A}$.

We next argue that any minimizer must belong to the relative interior of $\Xi_+ \times \mathcal{A}$. To this end, suppose that $\boldsymbol{x} \in \Xi_+$ and $\boldsymbol{\alpha} \in \mathcal{A}$ are given and such that $\alpha_i = 0$ for some i. We then define $\overline{\boldsymbol{\alpha}} := (\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}/2, \alpha_{i+1}/2, \alpha_{i+2}, \dots, \alpha_N)$ and $\overline{\boldsymbol{x}} := (x_0, \dots, x_{i-2}, x_{i-1} + x_i, 0, x_{i+1}, \dots, x_N)$ and observe that $C^V(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\alpha}}) = C^V(\boldsymbol{x}, \boldsymbol{\alpha})$. But Proposition 2.10 implies that $\overline{\boldsymbol{x}}$ cannot be optimal for $\overline{\boldsymbol{\alpha}}$ since $\overline{x}_i = 0$. In particular, $(\boldsymbol{x}, \boldsymbol{\alpha})$ cannot be optimal. Thus, the $\boldsymbol{\alpha}$-component of any minimizer must lie in the relative interior of $\mathcal{A}$. Finally, for $\boldsymbol{\alpha}$ in the relative interior of $\mathcal{A}$, Proposition 2.10 states that $\overline{\boldsymbol{x}}^*(\boldsymbol{\alpha})$ belongs to the relative interior of Ξ_+.

Now suppose that $(\boldsymbol{x}, \boldsymbol{\alpha})$ is a minimizer of C^V. Due to the preceding step, there must be Lagrange multipliers ν and λ such that $(\boldsymbol{x}, \boldsymbol{\alpha})$ is a critical point of $(\boldsymbol{y}, \boldsymbol{\beta}) \mapsto C^V(\boldsymbol{y}, \boldsymbol{\beta}) - \nu \sum_{i=0}^N y_i - \lambda \sum_{j=1}^N \beta_j$. The identity (43) thus again yields

$$\nu(1 - e^{-\alpha_{i+1}}) = F^{-1}(E_i + x_i) - e^{-\alpha_{i+1}} F^{-1}(E_{i+1}), \qquad i = 0, \dots, N-1, \tag{48}$$

and

$$\nu = F^{-1}(E_N + x_N). \tag{49}$$

Using the same argument as in the proof of Proposition 2.10, we have $\nu > 0$. Note that this can also be obtained by writing

$$-X_0 = \sum_{i=0}^N x_i = F(\nu) + \sum_{i=1}^N (1 - a_i) h_{1,a_i}^{-1}(\nu(1 - a_i)).$$

Indeed, the right-hand side is strictly increasing in ν (F and the functions h_{1,a_i}^{-1} are strictly increasing) and vanishes for $\nu = 0$, so $\nu > 0$.

Next, (44) gives

$$\lambda = E_j \sum_{n=j}^N \big[F^{-1}(E_n + x_n) - F^{-1}(E_n)\big] e^{-\sum_{k=j+1}^n \alpha_k}, \qquad j = 1, \dots, N. \tag{50}$$

We now rewrite the sum in (50) as follows:

$$\begin{aligned}
&\sum_{n=j}^{N}\big[F^{-1}(E_n+x_n)-F^{-1}(E_n)\big]e^{-\sum_{k=j+1}^{n}\alpha_k}\\
&\quad= -F^{-1}(E_j)\\
&\qquad+\big[F^{-1}(E_j+x_j)-F^{-1}(E_{j+1})e^{-\alpha_{j+1}}\big]+\dots\\
&\qquad+\big[F^{-1}(E_{N-1}+x_{N-1})-F^{-1}(E_N)e^{-\alpha_N}\big]e^{-\sum_{k=j+1}^{N-1}\alpha_k}\\
&\qquad+F^{-1}(E_N+x_N)e^{-\sum_{k=j+1}^{N}\alpha_k}.
\end{aligned}$$

Plugging in (48) and (49), simplifications occur and we get

$$\sum_{n=j}^{N}\big[F^{-1}(E_n+x_n)-F^{-1}(E_n)\big]e^{-\sum_{k=j+1}^{n}\alpha_k}=\nu-F^{-1}(E_j).$$

Plugging this back into (50) yields $\lambda=(\nu-F^{-1}(E_j))E_j$ for $j=1,\dots,N$. Solving this equation together with (48) for ν and λ implies that necessarily

$$\begin{aligned}
\nu &= \frac{F^{-1}(E_{i-1}+x_{i-1})-e^{-\alpha_i}F^{-1}(e^{-\alpha_i}(E_{i-1}+x_{i-1}))}{1-e^{-\alpha_i}},\\
\lambda &= e^{-\alpha_i}(E_{i-1}+x_{i-1})\frac{F^{-1}(E_{i-1}+x_{i-1})-F^{-1}(e^{-\alpha_i}(E_{i-1}+x_{i-1}))}{1-e^{-\alpha_i}},
\end{aligned}$$

for $i=1,\dots,N$. Lemma 3.3 (c) thus implies that

$$\alpha_1=\dots=\alpha_N \qquad\text{and}\qquad x_0=E_1+x_1=\dots=E_{N-1}+x_{N-1}.$$

This gives $\boldsymbol{\alpha}=\boldsymbol{\alpha}^*$. Moreover, (15) holds since $x_i=(1-a^*)x_0$ for $i=1,\dots,N-1$. We also get $E_i=a^*x_0$ for $i=1,\dots,N$. Note next that $x_N=-X_0-x_0-(N-1)(1-a^*)x_0$ and therefore $E_N+x_N=-X_0-N(1-a^*)x_0$. Equation (14) now follows from the fact that

$$F^{-1}(-X_0-N(1-a^*))=F^{-1}(E_N+x_N)=\frac{\partial C^V}{\partial x_N}(\boldsymbol{x},\boldsymbol{\alpha})=\nu=\frac{F^{-1}(x_0)-a^*F^{-1}(a^*x_0)}{1-a^*}.$$

This concludes the proof of the theorem. □

Proof of Corollary 2.12: The result follows immediately from Theorem 2.11. □

3.3 Proofs for reversion of price impact

The general strategy of the proof is similar to the one Section 3.2, although there are also some differences. We start with two lemmas on properties of the functions satisfying Assumption 2.15. Their conclusions are more important than Assumption 2.15 itself, as the validity of the conclusions of Lemmas 3.7 and 3.8 will imply the validity of Theorem 2.17.

Lemma 3.9 provides recursive identities for the gradient of our cost functional. These identities are needed to derive equations for critical points of the constraint optimization problem. The existence of such critical points is guaranteed by Lemma 3.11. Uniqueness, however, must be proved by another method as in Section 3.2, because the cost functional is no longer convex for price impact reversion.

Again, it will be enough to prove our results for $X_0 < 0$. The case for $X_0 > 0$ will then follow by the symmetry of the problem formulation. The case of round trips with $X_0 = 0$ will be analyzed by a limiting procedure.

Lemma 3.7. *Under Assumption 2.15, the following conclusions hold.*

(a) *$x \mapsto xf(x)$ is increasing on $\mathbb{R}$ (or, equivalently, $\tilde{F}$ is convex).*

(b) *For all $a \in (0,1)$, $x \mapsto af(ax)/f(x)$ is nondecreasing on $\mathbb{R}_+$ and nonincreasing on $\mathbb{R}_-$ and takes values in $(0,1)$.*

(c) *For all $x > 0$, $(0,1) \ni a \longmapsto \frac{1-a^2 f(ax)/f(x)}{1-af(ax)/f(x)}$ is increasing.*

(d) *For all $x > 0$, $(0,1) \ni a \longmapsto a^{-1}\frac{1-a^2 f(x)/f(x/a)}{1-af(x)/f(x/a)}$ is decreasing.*

Proof under Assumption 2.15 (a): (a) The derivative is positive since $xf'(x)/f(x) > -1$ by Assumption 2.15 (a).

(b) Since $x \mapsto xf(x)$ is increasing, $af(ax)/f(x) = [axf(ax)]/[xf(x)] \in (0,1)$. The derivative of $x \mapsto af(ax)/f(x)$ is equal to $[a^2 f'(ax)f(x) - af(ax)f'(x)]/f(x)^2$. It is nonnegative on $\mathbb{R}_+$ and nonpositive on $\mathbb{R}_-$ if and only if

$$\frac{af'(ax)}{f(ax)} \geq \frac{f'(x)}{f(x)} \quad \text{for } x \geq 0, \text{ and} \quad \frac{af'(ax)}{f(ax)} \leq \frac{f'(x)}{f(x)} \quad \text{for } x \leq 0.$$

These conditions hold as a direct consequence of (20).

(c) For a fixed $x \geq 0$, we set $\psi(a) = af(ax)/f(x)$, which takes values in $(0,1)$. We need to show that

$$\frac{d}{da}\frac{1-a\psi(a)}{1-\psi(a)} = \frac{(1-a)\psi'(a) - \psi(a)(1-\psi(a))}{(1-\psi(a))^2} > 0.$$

This condition holds if and only if

$$\frac{\psi'(a)}{\psi(a)} > \frac{1-\psi(a)}{1-a}.$$

It is thus sufficient to show that ψ'/ψ is nonincreasing, since then we would have

$$1-\psi(a) < \int_a^1 \frac{\psi'(u)}{\psi(u)}\,du \leq (1-a)\frac{\psi'(a)}{\psi(a)}.$$

This leads to requiring $\psi\psi'' - (\psi')^2 \leq 0$, which in turn leads to the following condition:

$$1 + (ax)^2\left(\frac{f'(ax)}{f(ax)}\right)^2 - (ax)^2\frac{f''(ax)}{f(ax)} \geq 0 \qquad \text{for } a \in (0,1).$$

The latter condition is ensured by Assumption (21), since $xf'(x)/f(x) \in (-1, 0]$ and thus

$$\left(\frac{xf'(x)}{f(x)}\right)^2 + \frac{xf'(x)}{f(x)} < 0.$$

(d) We fix $x > 0$ and let $\tilde{\psi}(a) := af(x)/f(x/a)$. We need to show that

$$\frac{d}{da}a^{-1}\frac{1-a\tilde{\psi}(a)}{1-\tilde{\psi}(a)} = \frac{\tilde{\psi}(a) - 1 + a\tilde{\psi}'(a)(1-a)}{a^2(1-\tilde{\psi}(a))^2} < 0.$$

This condition holds if and only if

$$a\tilde{\psi}'(a) < \frac{1 - \tilde{\psi}(a)}{1 - a}.$$

Hence it is enough to show that $a \mapsto a\tilde{\psi}'(a)$ is nondecreasing, because then we would have

$$1 - \tilde{\psi}(a) > \int_a^1 u\tilde{\psi}'(u)\, du \geq (1-a)a\tilde{\psi}'(a).$$

Some calculations lead to

$$\frac{d}{da}a\tilde{\psi}'(a) = \frac{1}{f(x/a)}\left(1 + \frac{x}{a}\frac{f'(x/a)}{f(x/a)} + 2\left(\frac{x}{a}\frac{f'(x/a)}{f(x/a)}\right)^2 - \left(\frac{x}{a}\right)^2 \frac{f''(x/a)}{f(x/a)}\right),$$

which is nonnegative by Assumption (21). □

Proof of Lemma 3.7 under Assumption 2.15 (a): Points (a) and (b) are trivial. To check (c) and (d), we have to show that

$$a \in (0,1),\ a \mapsto \frac{1 - a^{2+\alpha}}{1 - a^{1+\alpha}} \text{ and } a \mapsto \frac{a - a^{2+\alpha}}{1 - a^{2+\alpha}} = 1 - \frac{1-a}{1 - a^{2+\alpha}}$$

are increasing. It is however easy to check by derivating that $a \mapsto \frac{1-a^{\gamma}}{1-a^{\beta}}$ is increasing on $(0,1)$ when $0 < \beta < \gamma$, which gives the result. □

Lemma 3.8. *Under Assumption 2.15, the following conclusions hold.*

(a) *For each $a \in (0,1)$, the function $h_{P,a}(x) = x\frac{f(x/a)/a - af(x)}{f(x/a) - af(x)}$ is well-defined for $x \in \mathbb{R}$ and is strictly increasing.*

(b) *For all $a, b \in (0,1)$ and $\nu > 0$, we have the inequalities*

$$h_{P,a}^{-1}(\nu)/a > h_{P,b}^{-1}(\nu) \text{ and } h_{P,b}^{-1}(\nu) < \nu.$$

(c) *The function $H_P : (x,a) \in (0,\infty) \times (0,1) \mapsto \left(x\frac{f(x/a)/a - af(x)}{f(x/a) - af(x)}, -x^2 f(x)\frac{f(x/a)(1/a-1)}{f(x/a) - af(x)}\right)$ is one-to-one.*

Proof: (a) First let us observe that the denominator of $h_{P,a}$ is positive, since $x \mapsto xf(x)$ is increasing by Lemma 3.7 (a). We have

$$h_{P,a}(x) = x\left(1 + \frac{a^{-1} - 1}{1 - af(x)/f(x/a)}\right). \tag{51}$$

Again by Lemma 3.7, the fraction is positive and, as a function of x, nondecreasing on $\mathbb{R}_+$ and nonincreasing $\mathbb{R}_-$, which gives the result.

(b) It is clear from (51) that $h_{P,a}(x) > x$ for $x > 0$ and therefore $h_{P,a}^{-1}(x) < x$. Let us now consider $a, b \in (0,1)$, $\nu > 0$ and set $x' = h_{P,a}^{-1}(\nu)/a$, $x = h_{P,b}^{-1}(\nu)$. Then both x and x' are positive, and we need to show that $x' > x$. It follows that

$$\nu = x'\frac{f(x') - a^2 f(ax')}{f(x') - af(ax')} = x\frac{f(x/b)/b - bf(x)}{f(x/b) - bf(x)}.$$

Let us suppose by a way of contradiction that $x' \leq x$. Then, using Lemma 3.7 (b) and the fact that $u \in [0,1) \mapsto (1-au)/(1-u)$ is increasing, we get:

$$\frac{1-a^2f(ax)/f(x)}{1-af(ax)/f(x)} \geq \frac{1-a^2f(ax')/f(x')}{1-af(ax')/f(x')} \geq b^{-1}\frac{1-b^2f(x)/f(x/b)}{1-bf(x)/f(x/b)}.$$

Again by Lemma 3.7, the left-hand-side is increasing w.r.t a and the right-hand side is decreasing w.r.t. b. Moreover, both have the same limit,

$$\frac{2+xf'(x)/f(x)}{1+xf'(x)/f(x)},$$

when $a \uparrow 1$ and $b \uparrow 1$, which leads to a contradiction.

(c) Let $(a_1, y_1), (a_2, y_2) \in (0,1) \times (0, \infty)$ be such that $H_P(a_1, y_1) = H_P(a_2, y_2)$. By (51), we then have

$$\begin{cases} y_1(1+\gamma_1) = y_2(1+\gamma_2) \\ y_1^2 f(y_1)\gamma_1 = y_2^2 f(y_2)\gamma_2, \end{cases} \qquad \text{where } \gamma_i := \frac{(a_i^{-1}-1)f(y_i/a_i)}{f(y_i/a_i) - a_i f(y_i)} \text{ for } i = 1,2. \tag{52}$$

Let us assume for example that $\gamma_2 \leq \gamma_1$ and set $\eta = \gamma_2/\gamma_1 \in (0,1]$. Eliminating y_1 in (52) yields

$$\phi(\eta) := \left(\frac{1+\eta\gamma_1}{1+\gamma_1}\right)^2 f\left(y_2\frac{1+\eta\gamma_1}{1+\gamma_1}\right) - \eta f(y_2) = 0.$$

Since $x \mapsto xf(x)$ is increasing by Lemma 3.7 (a), we have

$$\eta \in (0,1), \phi(\eta) < \frac{1-\eta}{1+\gamma_1} f(y_2) < 0.$$

Thus, $\eta = 1$ is the only zero of $\phi(\eta)$. We may thus conclude that $\gamma_1 = \gamma_2$ and in turn that $y_1 = y_2$. Finally, the equality $\gamma_1 = \gamma_2$ leads to $a_1 = a_2$ due to Lemma 3.7 (d), since

$$1 + \gamma_i = a_i^{-1}\frac{1-a_i^2 f(y_i)/f(y_i/a_i)}{1-a_i f(y_i)/f(y_i/a_i)}.$$

□

In the Model with price impact reversion, we need to minimize the following cost functional:

$$C^P(x_0, \ldots, x_n, \boldsymbol{\alpha}) = \sum_{n=0}^{N} G(F(D_n) + x_n) - G(F(D_n)), \tag{53}$$

where $D_0 = 0$ and $D_n = e^{-\alpha_n}F^{-1}(x_{n-1} + F(D_{n-1}))$ for $1 \leq n \leq N$. By $\widehat{f}(x) = 1/f(F^{-1}(x))$ we denote again the derivative of F^{-1}.

Lemma 3.9. *We have the following recursive formula for $i = 0, \ldots, N-1$,*

$$\frac{\partial C^P}{\partial x_i} = F^{-1}\big(F(D_i) + x_i\big) + e^{-\alpha_{i+1}} f(D_{i+1})\widehat{f}\big(x_i + F(D_i)\big)\Big[\frac{\partial C^P}{\partial x_{i+1}} - D_{i+1}\Big]. \tag{54}$$

Moreover, for $j = 1, \ldots, N$,

$$\frac{\partial C^P}{\partial \alpha_j} = -D_j f(D_j)\left(\frac{\partial C^P}{\partial x_j} - D_j\right). \tag{55}$$

Proof: We have $D_1 = e^{-\alpha_1}F^{-1}(x_0)$ and $D_n = e^{-\alpha_n}F^{-1}(x_{n-1} + F(D_{n-1}))$ for $1 \le n \le N$. Thus, we obtain the following recursive relations between the derivatives of D_n with respect to x_i.

$$\begin{aligned}\frac{\partial D_n}{\partial x_i} &= 0 \qquad \text{for } i \ge n, \quad \frac{\partial D_n}{\partial x_{n-1}} = e^{-\alpha_n}\widehat{f}(x_{n-1} + F(D_{n-1})),\\ \frac{\partial D_n}{\partial x_i} &= e^{-\alpha_{i+1}} f(D_{i+1})\widehat{f}(x_i + F(D_i))\frac{\partial D_n}{\partial x_{i+1}} \qquad \text{for } 1 \le i \le n-1.\end{aligned}$$

Thus, by (53),

$$\begin{aligned}\frac{\partial C^P}{\partial x_i} &= F^{-1}(x_i + F(D_i)) + \sum_{n=i+1}^{N} f(D_n)\big[F^{-1}(x_n + F(D_n)) - D_n\big]\frac{\partial D_n}{\partial x_i} \qquad (56)\\ &= F^{-1}(x_i + F(D_i)) + e^{-\alpha_{i+1}} f(D_{i+1})\widehat{f}\big(x_i + F(D_i)\big)\big[F^{-1}(x_{i+1} + F(D_{i+1})) - D_{i+1}\big]\\ &\quad + e^{-\alpha_{i+1}} f(D_{i+1})\widehat{f}\big(x_i + F(D_i)\big)\sum_{n=i+2}^{N} f(D_n)\big[F^{-1}(x_n + F(D_n)) - D_n\big]\frac{\partial D_n}{\partial x_{i+1}}.\end{aligned}$$

By (56), the sum in the preceding line satisfies

$$\sum_{n=i+2}^{N} f(D_n)\big[F^{-1}(x_n + F(D_n)) - D_n\big]\frac{\partial D_n}{\partial x_{i+1}} = \frac{\partial C^P}{\partial x_{i+1}} - F^{-1}(x_{i+1} + F(D_{i+1})).$$

Hence,

$$\frac{\partial C^P}{\partial x_i} = F^{-1}(x_i + F(D_i)) + e^{-\alpha_{i+1}} f(D_{i+1})\widehat{f}(x_i + F(D_i))\Big[\frac{\partial C^P}{\partial x_{i+1}} - D_{i+1}\Big],$$

which is our formula (54).

As to (55), we use again the recursive scheme at the beginning of this proof to obtain formulas for the derivatives of D_n with respect to α_j:

$$\begin{aligned}\frac{\partial D_n}{\partial \alpha_j} &= 0 \qquad \text{for } j > n, \quad \frac{\partial D_n}{\partial \alpha_n} = -D_n,\\ \frac{\partial D_n}{\partial \alpha_j} &= -D_j f(D_j)\frac{\partial D_n}{\partial x_j} \qquad \text{for } 1 \le j \le n-1.\end{aligned}$$

We therefore obtain from (53):

$$\begin{aligned}\frac{\partial C^P}{\partial \alpha_i} &= \sum_{n=i}^{N} f(D_n)\big[F^{-1}(x_n + F(D_n)) - D_n\big]\frac{\partial D_n}{\partial \alpha_i}\\ &= -D_i f(D_i)[F^{-1}(x_i + F(D_i)) - D_i]\\ &\quad - \sum_{n=i+1}^{N} f(D_n) D_i f(D_i)\big[F^{-1}(x_n + F(D_n)) - D_n\big]\frac{\partial D_n}{\partial x_i}\\ &= -D_i f(D_i)\left(F^{-1}(x_i + F(D_i)) - D_i + \frac{\partial C^P}{\partial x_i} - F^{-1}(x_i + F(D_i))\right)\\ &= -D_i f(D_i)\left(\frac{\partial C^P}{\partial x_i} - D_i\right).\end{aligned}$$

Remark 3.10. A consequence of this lemma is that the optimal strategy given by [2] on the homogeneous time spacing grid $\mathcal{T}^*$ is a critical point for the minimization in $(\boldsymbol{x}, \boldsymbol{\alpha})$. Indeed, we have then $D_i = a^* F^{-1}(\xi_0^P)$ for any i, and therefore $\frac{\partial C^P}{\partial \alpha_i}$ does not depend on i. $\diamondsuit$

Lemma 3.11. *Assume that $\boldsymbol{\alpha} \in \mathcal{A}^*$. Then, $C^P(\boldsymbol{x}, \boldsymbol{\alpha}) \to \infty$ as $|\boldsymbol{x}| \to \infty$ under Assumption 2.15.*

Proof: Equation (53) yields

$$\begin{aligned} &C^P(\boldsymbol{x}, \boldsymbol{\alpha}) \\ &= \sum_{n=0}^{N} \tilde{F}\big(F^{-1}(F(D_n) + x_n)\big) - \tilde{F}(D_n) \\ &= \sum_{n=0}^{N-1} \Big[\tilde{F}\big(F^{-1}(F(D_n) + x_n)\big) - \tilde{F}\big(e^{-\alpha_{n+1}} F^{-1}(F(D_n) + x_n)\big)\Big] + \tilde{F}\big(F^{-1}(F(D_N) + x_N)\big). \end{aligned}$$

Let $\overline{a} = \max_{i=1,\dots,N} e^{-\alpha_i} < 1$. Since $x \mapsto x f(x)$ is increasing on $\mathbb{R}$, we have for $x \in \mathbb{R},\ a \in [0, \overline{a}]$,

$$\tilde{F}(x) - \tilde{F}(ax) = \int_{ax}^{x} y f(y) dy \geq \int_{\overline{a}x}^{x} y f(y) dy \geq \overline{a}(1 - \overline{a}) x^2 f(\overline{a}x) =: H(x).$$

Defining $T_2(\boldsymbol{x}) = \big(x_0, x_1 + F^{-1}(D_1), \dots, x_N + F^{-1}(D_N)\big)$, we thus get

$$C^P(\boldsymbol{x}, \boldsymbol{\alpha}) \geq H(|T_2(\boldsymbol{x})|_\infty).$$

From (20), $x \mapsto x f(x)$ is increasing and therefore $H(x) \to +\infty$ as $|x| \to +\infty$. It is therefore sufficient to have $T_2(\boldsymbol{x}) \to +\infty$ for $|\boldsymbol{x}| \to +\infty$. To this end, let $(\boldsymbol{x}^k)$ be a sequence such that the sequence $(T_2(\boldsymbol{x}^k))$ is bounded. We will show that $(\boldsymbol{x}^k)$ then must also be bounded. It is clear that the first coordinate x_0^k is bounded. Therefore, $F^{-1}(D_1^k)$ is also bounded, which in turn implies that the second coordinate of $(T_2(\boldsymbol{x}^k))$ is bounded. We then get that (x_1^k) is bounded. An easy induction on coordinates thus gives the desired result. $\square$

We are now in position to prove the main results for the Model with price impact reversion.

Proof of Proposition 2.16: Let us first assume $X_0 < 0$. We can assume without loss of generality that $\boldsymbol{\alpha} \in \mathcal{A}^*$, for otherwise we can simply merge two trades occurring at the same time into a single trade. If $\boldsymbol{x}$ is the minimizer of $C^P(\cdot, \boldsymbol{\alpha})$ on Ξ, then there must be a Lagrange multiplier ν such that $\boldsymbol{x}$ is a critical point of $\boldsymbol{y} \mapsto C^P(\boldsymbol{y}, \boldsymbol{\alpha}) - \nu \sum_{i=0}^{N} y_i$. Hence, (54) yields that

$$\nu = h_{P,a_{i+1}}(D_{i+1}), \qquad i = 0, \dots, N-1,$$

where $a_{i+1} = e^{-\alpha_{i+1}}$ and $h_{P,a}$ is defined as in Lemma 3.8. Since $D_{i+1} = a_{i+1} F^{-1}(x_i + F(D_i))$, we get with Lemma 3.8 that

$$x_0 = F(h_{P,a_1}^{-1}(\nu)/a_1),\ x_i = F(h_{P,a_{i+1}}^{-1}(\nu)/a_{i+1}) - F(h_{P,a_i}^{-1}(\nu)), \qquad i = 1, \dots, N-1.$$

For the last trade, we also get that $\nu = F^{-1}(x_N + F(D_N))$ and $x_N = F(\nu) - F(h_{P,a_N}^{-1}(\nu))$. Therefore, summing all the trades, we get:

$$-X_0 = F(\nu) + \sum_{i=1}^{N} \big[F(h_{P,a_i}^{-1}(\nu)/a_i) - F(h_{P,a_i}^{-1}(\nu))\big]. \tag{57}$$

Now let us observe that F is increasing on $\mathbb{R}$, and for any $a \in (0,1)$, $y \mapsto F(y/a) - F(y)$ is increasing (its derivative is positive by Lemma 3.7 (a)). Besides, F and $h_{P,a}^{-1}$ are increasing for any $a \in (0,1)$ and therefore ν is uniquely determined by the above equation. We have moreover $\nu > 0$ because the left-hand side vanishes when ν is equal to 0. This proves that there a unique critical point, which then is necessarily the global minimum of C^P by Lemma 3.11.

Next, $x_i > 0$ for $i = 0, \dots, N$, due to Lemma 3.7 and the fact that F is increasing.

Finally we consider the case $X_0 = 0$. As in the proof of Proposition 2.10, we can show that a round trip such that $C^P(\boldsymbol{x}, \boldsymbol{\alpha}) \le 0$ necessarily satisfies $C^P(\boldsymbol{x}, \boldsymbol{\alpha}) = 0$. Moreover for $\boldsymbol{\alpha} \in \mathcal{A}^*$, we see looking at the proof of Proposition 2.16 that $(0, \dots, 0)$ is the only critical point when $X_0 = 0$ since we necessarily have $\nu = 0$ by (57). Therefore, it is also the unique minimum of C^P by Lemma 3.11. □

Proof of Theorem 2.17: The existence of a minimizer $(\boldsymbol{\xi}^P, \boldsymbol{\alpha}^*)$ and the fact that it belongs to $\Xi_+ \times \mathcal{A}^*$ follow exactly as in the proof of Theorem 2.11.

Now suppose that $(\boldsymbol{x}, \boldsymbol{\alpha})$ is a minimizer of C^P for $X_0 < 0$. Due to the preceding step, there must be Lagrange multipliers $\nu, \lambda \in \mathbb{R}$ such that $(\boldsymbol{x}, \boldsymbol{\alpha})$ is a critical point of $(\boldsymbol{y}, \boldsymbol{\beta}) \mapsto C^P(\boldsymbol{y}, \boldsymbol{\beta}) - \nu \sum_{i=0}^N y_i - \lambda \sum_{j=1}^N \beta_j$.

From (54), we easily obtain that for $i = 1, \dots, N$,

$$\nu = \frac{e^{-\alpha_i} f(D_i)}{f(e^{\alpha_i} D_i)} [\nu - D_i] + e^{\alpha_i} D_i$$

and $\nu = F^{-1}(x_N + F(D_N))$ for the last trade. We then deduce from (55) that

$$\begin{aligned} \nu &= D_i \frac{e^{\alpha_i} f(e^{\alpha_i} D_i) - e^{-\alpha_i} f(D_i)}{f(e^{\alpha_i} D_i) - e^{-\alpha_i} f(D_i)} \\ \lambda &= -D_i^2 f(D_i) \frac{(e^{\alpha_i} - 1) f(e^{\alpha_i} D_i)}{f(e^{\alpha_i} D_i) - e^{-\alpha_i} f(D_i)}, \end{aligned}$$

i.e., $(\nu, \lambda) = H_P(D_i, a_i)$ with $a_i = e^{-\alpha_i}$. As in the proof of Proposition 2.16 we get (57), which (by our standing assumption $X_0 < 0$) ensures $\nu > 0$ and in turn $D_i > 0$ for $i = 1, \dots, N$. Due to Lemma 3.8, H_P is one-to-one on $(0, \infty) \times (0, 1)$, and therefore $\alpha_1 = \dots = \alpha_N$ and $D_1 = \dots = D_N$. Then, $D_1 = a^* F^{-1}(x_0)$. Since $D_{i+1} = a^* F^{-1}(x_i + F(D_i))$, we get $x_i = x_0 - F(D_i) = x_0 - F(a^* F^{-1}(x_0))$, and therefore $x_N = -X_0 - N x_0 + (N-1) F(a^* F^{-1}(x_0))$. Combining this with $\nu = F^{-1}(x_N + F(D_N))$, we get

$$F^{-1}\big(- X_0 - N[x_0 - F(a^* F^{-1}(x_0))]\big) = h_{P,a^*}(F^{-1}(x_0)).$$

We refer to [2, Lemma C.3] for the existence, uniqueness, and positivity of the solution x_0 of this equation. It follows that there is a unique critical point of C^P on $\Xi_+ \times \mathcal{A}^*$, which is necessarily the global minimum. □

Proof of Corollary 2.18: The result follows immediately from Proposition 2.16 and Theorem 2.17. □

4 Conclusion

We have introduced two variants of a market impact model in which price impact is a nonlinear function of volume impact and in which either volume or price impact reverts on an exponential scale. In both model variants, there are unique optimal strategies for the liquidation or acquisition of asset positions, when optimality is defined in terms of the minimization of the expected liquidation costs. Existence and structure of these strategies allows us to conclude that our market impact model admits neither price manipulation in the sense of Huberman and Stanzl [14] nor transaction-triggered price manipulation in the sense of Alfonsi, Schied, and Slynko [3].

Our optimal execution strategies turn out to be deterministic, because we are minimizing the *expected* execution costs. As argued by Almgren and Chriss [5, 6], trade execution strategies used in practice should also take volatility risk into account, which may lead to adaptive strategies. We refer to [18, 19]. For future research, it would also be interesting to allow certain model parameters to be random.

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